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Solution of a system of delay differential equations of multi pantograph type

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Abstract

A collocation method is proposed to obtain an approximate solution of a system of multi pantograph type delay differential equations with variable coefficients subject to the initial conditions. The general approach is that, first of all the solution of the system has been expanded according to First Boubaker polynomials (FBPs) basis. Then, by employing the matrix operations and collocation nodes, the original problem and the associated initial conditions are reduced to a nonlinear system. By solving such system, the unknown coefficients of the approximate solution can be determined. Convergence analysis of the proposed method has been proved. The presented method has been tested of three different examples. The computed results confirm the high accuracy of collocation method based on FBPs.

Keywords: System of delay differential equations; First Boubaker polynomials; Approximate solution; Collocation method; Matrix equation

AMS subject Classification: 34k06; 34k28; 41A10; 65L05

1. Introduction

In 1851, it was the first time that a device named pantograph was used in the construction of the electric locomotive which this name was originated from that time. Pantograph was modeled mathematically in 1971 [1]. Pantograph equations are one of the most prominent kinds of functional differential equations with proportional delay and often appear in many scientific models such as number theory, nonlinear dynamical systems, electrodynamics, quantum mechanics, population studies and etc.

We consider the system of delay differential equations of multi pantograph type in the following general form:

$$\begin{aligned}\sum_{i_2=1}^{M} J_{i_1,i_2}(x)\, y_{i_2}^{(1)}(x) = \sum_{i_2=1}^{M} R^{*}_{i_1,i_2}(x)\, y_{i_2}(x) + \\ \sum_{i_3=1}^{\wp}\sum_{i_2=1}^{M} P^{i_3}_{i_1,i_2}(x)\, y_{i_2}(\mu_{i_3}x) + \sum_{q=2}^{\gamma}\sum_{i_4=1}^{\varpi}\sum_{i_2=1}^{M} P^{*\,i_4}_{i_1,i_2}(x)\,(y_{i_2}(\sigma_{i_4}x))^{q} + f_{i_1}(x), \\ M \geq 1,\ \ i_1 = 1\,(1)\,M,\ \ \wp,\ \varpi \in \mathbb{N},\ \ \gamma \in \mathbb{N} - \{1\},\ \ x \in I = [0,\ 1],\end{aligned} \tag{1}$$

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subject to the initial conditions

$$y_{i_1}(0) = \lambda_{i_1}, \quad i_1 = 1\,(1)\,M, \tag{2}$$

where $\{y_{i_1}(x)\}_{i_1=1}^{M}$ are the unknown functions to be determined, μ_{i_3}, $\sigma_{i_4} \in (0, 1)$ and λ_{i_1} are suitable constants. In addition, $J_{i_1,i_2}(x)$, $R^*_{i_1,i_2}(x)$, $P^{i_3}_{i_1,i_2}(x)$, $P^{*\,i_4}_{i_1,i_2}(x)$ and $f_{i_1}(x)$ are smooth bounded real functions on the interval I.

Liu and Li [2] proved that the multi pantograph equation

$$y^{(1)}(x) = \lambda\, y(x) + \sum_{i_5=1}^{l} q_{i_5} y(\mu_{i_5} x), \qquad y(0) = y_0,$$

has an unique analytic solution for any λ, $q_1, q_2, \ldots, q_l \in \mathbb{C}$ and this solution is asymptotically stable under the following conditions

$$\mathrm{Re}\lambda < 0 \quad \text{and} \quad \sum_{i_5=1}^{l} \left|q_{i_5}\right| < |\lambda|.$$

Many studies were carried out on the approximate solution of mentioned equation in one dimensional case. In this section, some of the studies are reviewed, They include application of the reproducing kernel space method, Bessel polynomial approach, model reduction approach, the enhanced multi-stage homotopy perturbation method and multi-step technique with differential transform method for delay differential equations (cf. [3-7]). The method based on Chebyshev polynomials, shifted Legendre approximation method, ortho exponential polynomial approach, perturbation-iteration algorithms and the Adomian decomposition method for pantograph type delay differential equations (cf. [8-12]).

In addition, Bessel collocation method, collocation method based on Bernoulli operational matrix, Jacobi rational-Gauss collocation method, high order stable Runge-Kutta method and shifted orthonormal Bernstein polynomials method have been introduced for solving generalized pantograph equations (cf. [13-17]).
For a comprehensive review of the methods for solving Eqs. (1) and (2), we refer the interested reader to the references [18-33].

There are few studies in the literature to solve Eqs. (1) and (2) in the case $M = 2, 3$ which are Bessel polynomial bases method [34, 38], Taylor collocation method [35], Euler matrix method [36] and method based on combination of Laplace transform and Adomian decomposition method [37], that they were studied in their nonlinear case only in the reference [37].

The purpose of the current study is to approximate the solution of Eqs. (1) and (2) by the truncated series as shown below:

$$y_{i_1}(x) \simeq y_{i_1,N}(x) = \sum_{i=0}^{N} \bar{y}_{i_1,i}\, B_i(x), \quad i_1 = 1\,(1)\,M, \quad x \in I, \tag{3}$$

in which N is selected as any positive integer in such a manner that $N \geq 1$, $\{B_i(x)\}_{i=0}^{N}$ shows that the FBPs of degree i and $\{\bar{y}_{i_1,i}\}_{i_1=1,\,i=0}^{M,N}$ are the unknown Boubaker coefficients which should be determined.

Nowadays FBPs have been used as a powerful tool for solving mixed linear integro-differential-difference equations and the linear pantograph equation with proportional delay (cf. [39, 40]).

To achieve the above mentioned goal, Eqs. (1) and (2) are reduced to the following matrix forms

$$\mathbf{J}(x)\,\mathbf{y}^{(1)}(x) = \mathbf{R}^*(x)\,\mathbf{y}(x) + \sum_{i_3=1}^{\wp} \mathbf{P}_{i_3}(x)\,\mathbf{y}(\mu_{i_3} x) + \sum_{q=2}^{\gamma} \sum_{i_4=1}^{\varpi} \mathbf{P}^*_{i_4}(x)\,(\mathbf{y}(\sigma_{i_4} x))^q + \mathbf{f}(x), \quad x \in I, \tag{4}$$

subject to the initial condition

$$\mathbf{y}(0) = \lambda, \tag{5}$$

where

$$\mathbf{J}(x) = \begin{bmatrix} J_{1,1}(x) & J_{1,2}(x) & \dots & J_{1,M}(x) \\ J_{2,1}(x) & J_{2,2}(x) & \dots & J_{2,M}(x) \\ \vdots & \vdots & \ddots & \vdots \\ J_{M,1}(x) & J_{M,2}(x) & \dots & J_{M,M}(x) \end{bmatrix}, \quad \mathbf{R}^*(x) = \begin{bmatrix} R^*_{1,1}(x) & R^*_{1,2}(x) & \dots & R^*_{1,M}(x) \\ R^*_{2,1}(x) & R^*_{2,2}(x) & \dots & R^*_{2,M}(x) \\ \vdots & \vdots & \ddots & \vdots \\ R^*_{M,1}(x) & R^*_{M,2}(x) & \dots & R^*_{M,M}(x) \end{bmatrix},$$

$$\mathbf{P}_{i_3}(x) = \begin{bmatrix} P^{i_3}_{1,1}(x) & P^{i_3}_{1,2}(x) & \dots & P^{i_3}_{1,M}(x) \\ P^{i_3}_{2,1}(x) & P^{i_3}_{2,2}(x) & \dots & P^{i_3}_{2,M}(x) \\ \vdots & \vdots & \ddots & \vdots \\ P^{i_3}_{M,1}(x) & P^{i_3}_{M,2}(x) & \dots & P^{i_3}_{M,M}(x) \end{bmatrix}, \quad \mathbf{P}^*_{i_4}(x) = \begin{bmatrix} P^{*\,i_4}_{1,1}(x) & P^{*\,i_4}_{1,2}(x) & \dots & P^{*\,i_4}_{1,M}(x) \\ P^{*\,i_4}_{2,1}(x) & P^{*\,i_4}_{2,2}(x) & \dots & P^{*\,i_4}_{2,M}(x) \\ \vdots & \vdots & \ddots & \vdots \\ P^{*\,i_4}_{M,1}(x) & P^{*\,i_4}_{M,2}(x) & \dots & P^{*\,i_4}_{M,M}(x) \end{bmatrix},$$

$$\mathbf{y}^{(1)}(x) = \begin{bmatrix} y^{(1)}_1(x) \\ y^{(1)}_2(x) \\ \vdots \\ y^{(1)}_M(x) \end{bmatrix}, \quad \mathbf{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_M(x) \end{bmatrix}, \quad \mathbf{y}(\mu_{i_3}x) = \begin{bmatrix} y_1(\mu_{i_3}x) \\ y_2(\mu_{i_3}x) \\ \vdots \\ y_M(\mu_{i_3}x) \end{bmatrix}, \quad (\mathbf{y}(\sigma_{i_4}x))^q = \begin{bmatrix} (y_1(\sigma_{i_4}x))^q \\ (y_2(\sigma_{i_4}x))^q \\ \vdots \\ (y_M(\sigma_{i_4}x))^q \end{bmatrix},$$

$$\mathbf{f}(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_M(x) \end{bmatrix}, \quad \lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_M \end{bmatrix}.$$

Then the matrix equations correspond to the unknown vector functions in the above equation were achieved. All of these equations are based on the unknown Boubaker coefficient vector. A new matrix equation yield by substituting obtained matrix equations in the Eqs. (4) and (5). Finally, the collocation points are used in order to convert the resulting equation to a system of nonlinear algebraic equations. After solving this system, unknown Boubaker's coefficient vector and therefore unknown functions can be determined.

2. FBPs and some of their basic properties

In 2006, Karem Boubaker et al. [41] introduced FBPs for the first time. The motivation of this presentation was a tool for solving heat transfer equation. The definition offered by them is as follows:

Definition: The FBPs of degree n are defined on I as (cf. [40])

$$B_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n-k}{k} \frac{n-4k}{n-k}\, x^{n-2k}, \quad n \geq 1, \tag{6}$$

here, $B_0(x) = 1$ and $\left[\frac{n}{2}\right]$ shows the greatest integer in $\frac{n}{2}$.

For FBPs, the recursive relation is established as follows

$$B_{n+1}(x) = x\, B_n(x) - B_{n-1}(x), \quad n \geq 2,$$

in which $B_0(x) = 1$, $B_1(x) = x$ and $B_2(x) = x^2 + 2$. If $A_n(x)$, $O_n(x)$ and $C_n(x)$ are defined as

$$A_n(x) = \left(x^2 - 1\right)\left(3\,n\,x^2 + n - 2\right), \quad O_n(x) = 3\,x\left(n\,x^2 + 3\,n - 2\right), \quad C_n(x) = -n\left(3\,n^2x^2 + n^2 - 6\,n + 8\right),$$

then FBPs satisfy in the following second order differential equation

$$A_n(x)\, y^{(2)}(x) + O_n(x)\, y^{(1)}(x) - C_n(x)\, y(x) = 0.$$

These polynomials also have the generating function

$$f_B(t,\ x) = \frac{1+3\,x^2}{1+x\,(x-t)}.$$

If Eq. (6) is denoted by

$$B_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} a_{n,k}\, x^{n-2k}, \quad n = 0,\ 1,\ \ldots,\ N,$$

then the FBPs can also be expressed in terms of vector-matrix form, using vector-matrix notation

$$\mathbf{B}(x) = \mathbf{A}\,\mathbf{T}(x) \quad \text{or} \quad \mathbf{B}^T(x) = \mathbf{T}^T(x)\,\mathbf{A}^T, \tag{7}$$

in which $\mathbf{B}(x)$ and $\mathbf{T}(x)$ are $(N+1)\times 1$ vector functions of FBPs and Taylor polynomials respectively as

$$\mathbf{B}(x) = [B_0(x),\ B_1(x),\ \ldots,\ B_N(x)]^T, \quad \mathbf{T}(x) = \left[1,\ x,\ \ldots,\ x^N\right]^T, \quad x \in I,$$

and $\mathbf{A}$ is $(N+1)\times(N+1)$ lower triangular matrix such that if N is odd,

$$\mathbf{A} = \begin{bmatrix} a_{0,0} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a_{1,0} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_{2,1} & 0 & a_{2,0} & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a_{3,1} & 0 & a_{3,0} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ a_{N-1,\frac{N-1}{2}} & 0 & a_{N-1,\frac{N-3}{2}} & 0 & \cdots & a_{N-1,1} & 0 & a_{N-1,0} & 0 \\ 0 & a_{N,\frac{N-1}{2}} & 0 & a_{N,\frac{N-3}{2}} & \cdots & 0 & a_{N,1} & 0 & a_{N,0} \end{bmatrix},$$

if N is even,

$$\mathbf{A} = \begin{bmatrix} a_{0,0} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a_{1,0} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_{2,1} & 0 & a_{2,0} & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a_{3,1} & 0 & a_{3,0} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & a_{N-1,\frac{N-2}{2}} & 0 & a_{N-1,\frac{N-4}{2}} & \cdots & a_{N-1,1} & 0 & a_{N-1,0} & 0 \\ a_{N,\frac{N}{2}} & 0 & a_{N,\frac{N-2}{2}} & 0 & \cdots & 0 & a_{N,1} & 0 & a_{N,0} \end{bmatrix}.$$

It is easy to show that the components $\{a_{n,k}\}_{n=0,\,k=0}^{N,\left[\frac{n}{2}\right]}$ of $\mathbf{A}$ can be calculated by applying the following relations:

$$\begin{aligned} a_{n,0} &= 1, \\ a_{n,1} &= -(n-4), \\ a_{n,k} &= (-1)^k \tfrac{n-4k}{k!} \prod_{j=k+1}^{2k-1} (n-j), \quad k = 2,\ 3,\ \ldots,\ \left[\tfrac{n}{2}\right]. \end{aligned}$$

3. The matrix equations corresponding to unknown vector functions

In this section, the matrix equations corresponding to the unknown vector functions in the Eq. (4) are provided.

3.1. *The matrix equation corresponding to* $\mathbf{y}(x)$

The truncated Boubaker series (3) can be written in the matrix forms as:

$$y_{i_1}(x) \simeq \mathbf{Y}_{i_1}^T \mathbf{B}(x) = \mathbf{B}^T(x)\mathbf{Y}_{i_1}, \quad i_1 = 1\,(1)\,M, \tag{8}$$

in which $\{\mathbf{Y}_{i_1}\}_{i_1=1}^{M}$ are $(N+1) \times 1$ unknown Boubaker coefficient vectors and are defined as follows:

$$\mathbf{Y}_{i_1} = [\bar{y}_{i_1,0}, \bar{y}_{i_1,1}, \ldots, \bar{y}_{i_1,N}]^T, \quad i_1 = 1\,(1)\,M.$$

By placing Eq. (7) in the matrix forms (8), the following equation is derived at:

$$y_{i_1}(x) \simeq \mathbf{T}^T(x)\,\mathbf{A}^T\,\mathbf{Y}_{i_1}, \quad i_1 = 1\,(1)\,M. \tag{9}$$

Hence, the matrix equation corresponding to $\mathbf{y}(x)$ can be expressed by the following equation:

$$\mathbf{y}(x) \simeq \hat{\mathbf{T}}(x)\,\hat{\mathbf{A}}\,\mathbf{Y}, \tag{10}$$

where $\hat{\mathbf{T}}(x)$, $\hat{\mathbf{A}}$ and $\mathbf{Y}$ are of $M \times M(N+1)$, $M(N+1) \times M(N+1)$ and $M(N+1) \times 1$ dimensional and are defined as:

$$\hat{\mathbf{T}}(x) = \mathrm{Diag}\left(\mathbf{T}^T(x), \mathbf{T}^T(x), \ldots, \mathbf{T}^T(x)\right), \quad \hat{\mathbf{A}} = \mathrm{Diag}\left(\mathbf{A}^T, \mathbf{A}^T, \ldots, \mathbf{A}^T\right), \quad \mathbf{Y} = \begin{bmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \\ \vdots \\ \mathbf{Y}_M \end{bmatrix}.$$

3.2. *The matrix equation corresponding to* $\mathbf{y}^{(1)}(x)$

By differentiation both sides of Eq. (9) with respect to x, the following relation is derived at:

$$y_{i_1}^{(1)}(x) \simeq (\mathbf{T}^{(1)}(x))^T\mathbf{A}^T\,\mathbf{Y}_{i_1}, \quad i_1 = 1\,(1)\,M. \tag{11}$$

The relation between the vector $\mathbf{T}(x)$ and its derivative $\mathbf{T}^{(1)}(x)$ is expressed by the following equation:

$$\mathbf{T}^{(1)}(x) = \mathbf{D}_t\mathbf{T}(x) \quad \Rightarrow \quad (\mathbf{T}^{(1)}(x))^T = \mathbf{T}^T(x)\,\mathbf{D}_t^T, \tag{12}$$

where the $(N+1)\times(N+1)$ matrix $\mathbf{D}_t$ is the operational matrix of differentiation associated with the Taylor polynomials and is given by [40]:

$$\mathbf{D}_t = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & N & 0 \end{bmatrix}.$$

applying Eq. (12) in (11), yields:

$$y_{i_1}^{(1)}(x) \simeq \mathbf{T}^T(x)\,\mathbf{D}_t^T\,\mathbf{A}^T\,\mathbf{Y}_{i_1}, \quad i_1 = 1\,(1)\,M.$$

So, the matrix equation corresponding to $\mathbf{y}^{(1)}(x)$ can be obtained as:

$$\mathbf{y}^{(1)}(x) \simeq \hat{\mathbf{T}}(x)\,\hat{\mathbf{D}}_t\,\hat{\mathbf{A}}\,\mathbf{Y}, \tag{13}$$

where $M(N+1) \times M(N+1)$ block diagonal matrix $\hat{\mathbf{D}}_t$ is found by $\hat{\mathbf{D}}_t = \mathrm{Diag}\left(\mathbf{D}_t^T, \mathbf{D}_t^T, \ldots, \mathbf{D}_t^T\right)$.

3.3. The matrix equation corresponding to $\mathbf{y}(\mu_{i_3} x)$

By substituting quantities $\mu_{i_3} x$ instead of x in Eq. (9), the following matrix form can be obtained

$$y_{i_1}(\mu_{i_3} x) \simeq \mathbf{T}^T(\mu_{i_3} x)\,\mathbf{A}^T\,\mathbf{Y}_{i_1}, \quad i_1 = 1\,(1)\,M, \;\; i_3 = 1\,(1)\,\wp. \tag{14}$$

It is easily shown that the connection between the vectors $\mathbf{T}(\mu_{i_3} x)$ and $\mathbf{T}(x)$ is as follow:

$$\mathbf{T}(\mu_{i_3} x) = \mathbf{Z}_{\mu_{i_3}}\,\mathbf{T}(x) \quad \Rightarrow \mathbf{T}^T(\mu_{i_3} x) = \mathbf{T}^T(x)\,\mathbf{Z}_{\mu_{i_3}}, \quad i_3 = 1\,(1)\,\wp, \tag{15}$$

such that $\mathbf{Z}_{\mu_{i_3}} = \text{diag}\left((\mu_{i_3})^0,\, (\mu_{i_3})^1,\, \ldots,\, (\mu_{i_3})^N\right)$, $i_3 = 1\,(1)\,\wp$. Incorporating Eq. (15) in (14), yields:

$$y_{i_1}(\mu_{i_3} x) \simeq \mathbf{T}^T(x)\,\mathbf{Z}_{\mu_{i_3}}\,\mathbf{A}^T\,\mathbf{Y}_{i_1}, \quad i_1 = 1\,(1)\,M, \;\; i_3 = 1\,(1)\,\wp.$$

Thus, the matrix equation corresponding to the vector function $\mathbf{y}(\mu_{i_3} x)$ can be found via the following relation:

$$\mathbf{y}(\mu_{i_3} x) \simeq \hat{\mathbf{T}}(x)\,\hat{\mathbf{Z}}_{\mu_{i_3}}\,\hat{\mathbf{A}}\,\mathbf{Y}, \quad i_3 = 1\,(1)\,\wp, \tag{16}$$

where the $M(N+1)\times M(N+1)$ block diagonal matrix $\hat{\mathbf{Z}}_{\mu_{i_3}}$ is found by $\hat{\mathbf{Z}}_{\mu_{i_3}} = \text{Diag}\left(\mathbf{Z}_{\mu_{i_3}},\, \mathbf{Z}_{\mu_{i_3}},\, \ldots,\, \mathbf{Z}_{\mu_{i_3}}\right)$, $i_3 = 1\,(1)\,\wp$.

3.4. The matrix equation corresponding to $(\mathbf{y}(\sigma_{i_4} x))^q$

Similar to the process that studied in subsection 3.3, the following approximations are derived for the functions $y_{i_1}(\sigma_{i_4} x)$

$$y_{i_1}(\sigma_{i_4} x) \simeq \mathbf{T}^T(x)\,\mathbf{Z}_{\sigma_{i_4}}\,\mathbf{A}^T\,\mathbf{Y}_{i_1}, \quad i_1 = 1\,(1)\,M, \;\; i_4 = 1\,(1)\,\varpi,$$

such that $\mathbf{Z}_{\sigma_{i_4}} = \text{diag}\left((\sigma_{i_4})^0,\, (\sigma_{i_4})^1,\, \ldots,\, (\sigma_{i_4})^N\right)$, $i_4 = 1\,(1)\,\varpi$.
So,

$$(y_{i_1}(\sigma_{i_4} x))^q \simeq \left(\mathbf{T}^T(x)\,\mathbf{Z}_{\sigma_{i_4}}\,\mathbf{A}^T\,\mathbf{Y}_{i_1}\right)^{q-1}\left(\mathbf{T}^T(x)\,\mathbf{Z}_{\sigma_{i_4}}\,\mathbf{A}^T\,\mathbf{Y}_{i_1}\right), \quad i_1 = 1(1)M, \;\; i_4 = 1(1)\varpi, \;\; q = 2(1)\gamma.$$

Consequently, the matrix equation corresponding to the vector function $(\mathbf{y}(\sigma_{i_4} x))^q$ can be expressed as follows:

$$(\mathbf{y}(\sigma_{i_4} x))^q = \left(\hat{\mathbf{T}}(x)\,\hat{\mathbf{Z}}_{\sigma_{i_4}}\,\hat{\mathbf{A}}\,\hat{\mathbf{Y}}\right)^{q-1}\hat{\mathbf{T}}(x)\,\hat{\mathbf{Z}}_{\sigma_{i_4}}\,\hat{\mathbf{A}}\,\mathbf{Y}, \quad i_4 = 1\,(1)\,\varpi, \;\; q = 2\,(1)\,\gamma, \tag{17}$$

in which the $M(N+1)\times M(N+1)$ and $M(N+1)\times M$ block matrices $\hat{\mathbf{Z}}_{\sigma_{i_4}}$ and $\hat{\mathbf{Y}}$ are found by $\hat{\mathbf{Z}}_{\sigma_{i_4}} = \text{Diag}\left(\mathbf{Z}_{\sigma_{i_4}},\, \mathbf{Z}_{\sigma_{i_4}},\, \ldots,\, \mathbf{Z}_{\sigma_{i_4}}\right)$ and $\hat{\mathbf{Y}} = \text{Diag}\,(\mathbf{Y}_1,\, \mathbf{Y}_2,\, \ldots,\, \mathbf{Y}_M)$.

4. The collocation method

Collocation method to approximate the solution of Eqs. (1) and (2), is presented in this section. The study therefore, considers the matrix equations corresponding to aforesaid system as shown in Eqs. (4) and (5). In order to solve these equations with respect to unknown function $\mathbf{y}(x)$, the matrix forms (13), (10), (16) and (17) in accordance with the vector functions $\mathbf{y}^{(1)}(x)$, $\mathbf{y}(x)$, $\mathbf{y}(\mu_{i_3} x)$ and $(\mathbf{y}(\sigma_{i_4} x))^q$ are substituted in Eq. (4) and the following matrix equation can be derived in terms of the unknown coefficients vector $\mathbf{Y}$.

$$\begin{aligned}\mathbf{J}(x)\,\hat{\mathbf{T}}(x)\,\hat{\mathbf{D}}_t\,\hat{\mathbf{A}}\,\mathbf{Y} = \mathbf{R}^*(x)\,\hat{\mathbf{T}}(x)\,\hat{\mathbf{A}}\,\mathbf{Y} + \sum_{i_3=1}^{\wp}\mathbf{P}_{i_3}(x)\,\hat{\mathbf{T}}(x)\,\hat{\mathbf{Z}}_{\mu_{i_3}}\,\hat{\mathbf{A}}\,\mathbf{Y} + \\ \sum_{q=2}^{\gamma}\sum_{i_4=1}^{\varpi}\mathbf{P}^*_{i_4}(x)\left(\hat{\mathbf{T}}(x)\,\hat{\mathbf{Z}}_{\sigma_{i_4}}\,\hat{\mathbf{A}}\,\hat{\mathbf{Y}}\right)^{q-1}\hat{\mathbf{T}}(x)\,\hat{\mathbf{Z}}_{\sigma_{i_4}}\,\hat{\mathbf{A}}\,\mathbf{Y} + \mathbf{f}(x), \quad x \in I.\end{aligned} \tag{18}$$

To calculate the unknown coefficients vector $\mathbf{Y}$, the collocation points $x_\alpha = \frac{\alpha}{N}$, $\alpha = 0\,(1)\,N$, are used in the matrix relation (18). The resulting matrix equation system can be reduced in the following form:

$$\mathbf{J}\,\hat{\mathbf{T}}\,\hat{\mathbf{D}}_t\,\hat{\mathbf{A}}\,\mathbf{Y} = \mathbf{R}^*\,\hat{\mathbf{T}}\,\hat{\mathbf{A}}\,\mathbf{Y} + \sum_{i_3=1}^{\wp}\mathbf{P}_{i_3}\,\hat{\mathbf{T}}\,\hat{\mathbf{Z}}_{\mu_{i_3}}\,\hat{\mathbf{A}}\,\mathbf{Y} + \sum_{q=2}^{\gamma}\sum_{i_4=1}^{\varpi}\mathbf{P}^*_{i_4}\left(\hat{\hat{\mathbf{T}}}\,\hat{\hat{\mathbf{Z}}}_{\sigma_{i_4}}\,\hat{\hat{\mathbf{A}}}\,\hat{\hat{\mathbf{Y}}}\right)^{q-1}\hat{\mathbf{T}}\,\hat{\mathbf{Z}}_{\sigma_{i_4}}\,\hat{\mathbf{A}}\,\mathbf{Y} + \mathbf{F}, \tag{19}$$

where

$$\mathbf{J} = \mathrm{Diag}\ (\mathbf{J}(x_0),\, \mathbf{J}(x_1),\, \ldots,\, \mathbf{J}(x_N)),\quad \hat{\mathbf{T}}(x_\alpha) = \mathrm{Diag}\left(\mathbf{T}^T(x_\alpha),\, \mathbf{T}^T(x_\alpha),\, \ldots,\, \mathbf{T}^T(x_\alpha)\right),$$
$$\mathbf{R}^* = \mathrm{Diag}\ (\mathbf{R}^*(x_0),\, \mathbf{R}^*(x_1),\, \ldots,\, \mathbf{R}^*(x_N)),\quad \hat{\hat{\mathbf{T}}} = \mathrm{Diag}\left(\hat{\mathbf{T}}(x_0),\, \hat{\mathbf{T}}(x_1),\, \ldots,\, \hat{\mathbf{T}}(x_N)\right),$$
$$\mathbf{P}_{i_3} = \mathrm{Diag}\ (\mathbf{P}_{i_3}(x_0),\, \mathbf{P}_{i_3}(x_1),\, \ldots,\, \mathbf{P}_{i_3}(x_N)),\quad \hat{\hat{\mathbf{Z}}}_{\sigma_{i_4}} = \mathrm{Diag}\left(\hat{\mathbf{Z}}_{\sigma_{i_4}},\, \hat{\mathbf{Z}}_{\sigma_{i_4}},\, \ldots,\, \hat{\mathbf{Z}}_{\sigma_{i_4}}\right),$$
$$\mathbf{P}^*_{i_4} = \mathrm{Diag}\left(\mathbf{P}^*_{i_4}(x_0),\, \mathbf{P}^*_{i_4}(x_1),\, \ldots,\, \mathbf{P}^*_{i_4}(x_N)\right),\quad \hat{\hat{\mathbf{A}}} = \mathrm{Diag}\left(\hat{\mathbf{A}},\, \hat{\mathbf{A}},\, \ldots,\, \hat{\mathbf{A}}\right),$$
$$\hat{\mathbf{T}} = \begin{bmatrix} \hat{\mathbf{T}}(x_0) \\ \hat{\mathbf{T}}(x_1) \\ \vdots \\ \hat{\mathbf{T}}(x_N) \end{bmatrix}, \quad ,\mathbf{F} = \begin{bmatrix} \mathbf{f}(x_0) \\ \mathbf{f}(x_1) \\ \vdots \\ \mathbf{f}(x_N) \end{bmatrix}, \quad ,\hat{\hat{\mathbf{Y}}} = \mathrm{Diag}\left(\hat{\mathbf{Y}},\, \hat{\mathbf{Y}},\, \ldots,\, \hat{\mathbf{Y}}\right).$$

It is noteworthy that the matrices $\hat{\mathbf{T}}$, $\hat{\mathbf{T}}(x_\alpha)$, $\hat{\hat{\mathbf{T}}}$, $\hat{\hat{\mathbf{Z}}}_{\sigma_{i_4}}$, $\hat{\hat{\mathbf{A}}}$, $\hat{\hat{\mathbf{Y}}}$ and $\mathbf{F}$ are of $M(N+1)\times M(N+1)$, $M\times M(N+1)$, $M(N+1)\times M(N+1)^2$, $M(N+1)^2\times M(N+1)^2$, $M(N+1)^2\times M(N+1)^2$, $M(N+1)^2\times M(N+1)$ and $M(N+1)\times 1$ dimensional, respectively. In addition, all matrices $\mathbf{J}$, $\mathbf{R}^*$, $\mathbf{P}_{i_3}$, $\mathbf{P}^*_{i_4}$ are of dimension $M(N+1)\times M(N+1)$.
Eq. (19) can be summarized in the following compact form

$$\mathbf{W}\,\mathbf{Y} = \mathbf{F},$$

in which

$$\mathbf{W} = \left(\mathbf{J}\,\hat{\mathbf{T}}\,\hat{\mathbf{D}}_t - \mathbf{R}^*\,\hat{\mathbf{T}} - \sum_{i_3=1}^{\wp} \mathbf{P}_{i_3}\,\hat{\mathbf{T}}\,\hat{\mathbf{Z}}_{\mu_{i_3}} - \sum_{q=2}^{\gamma}\sum_{i_4=1}^{\varpi} \mathbf{P}^*_{i_4}\left(\hat{\hat{\mathbf{T}}}\,\hat{\hat{\mathbf{Z}}}_{\sigma_{i_4}}\,\hat{\hat{\mathbf{A}}}\,\hat{\hat{\mathbf{Y}}}\right)^{q-1}\hat{\mathbf{T}}\,\hat{\mathbf{Z}}_{\sigma_{i_4}}\right)\hat{\mathbf{A}}. \tag{20}$$

To create a matrix equation corresponding to the initial condition (5), the zero point is replaced instead of x in Eq. (10) and the resulting equation is substituted in Eq. (5). Accordingly, the matrix form of the initial condition is introduced as follows

$$\hat{\mathbf{T}}(0)\,\hat{\mathbf{A}}\,\mathbf{Y} = \lambda.$$

If matrix $\mathbf{U}$ is defined as

$$\mathbf{U} = \hat{\mathbf{T}}(0)\,\hat{\mathbf{A}},$$

then the matrix equation for the initial condition can be summarized by the following compact form

$$\mathbf{U}\,\mathbf{Y} = \lambda.$$

However, in order to apply initial condition (5) in Eq. (4), the rows of matrices $\mathbf{U}$ and λ are replaced with the M rows of matrices $\mathbf{W}$ and $\mathbf{F}$, respectively. Generally, the following single matrix equation can be obtained.

$$\mathbf{W}^*\,\mathbf{Y} - \mathbf{F}^* = \mathbf{0}. \tag{21}$$

The strategy is based on the fact that if the matrix $\mathbf{W}$ is singular, then the rows that have the same factor or all zeros are replaced. Otherwise, it is more comfortable that the aforementioned displacements occur in the last rows. If the latter is done, the rearrangement matrices $\mathbf{W}^*$ and $\mathbf{F}^*$ will be in the following forms:

$$\mathbf{W}^* = \begin{bmatrix} w_{1,1} & w_{1,2} & \cdots & w_{1,M(N+1)} \\ w_{2,1} & w_{2,2} & \cdots & w_{2,M(N+1)} \\ \vdots & \vdots & \ddots & \vdots \\ w_{MN,1} & w_{MN,2} & \cdots & w_{MN,M(N+1)} \\ u_{1,1} & u_{1,2} & \cdots & u_{1,M(N+1)} \\ u_{2,1} & u_{2,2} & \cdots & u_{2,M(N+1)} \\ \vdots & \vdots & \ddots & \vdots \\ u_{M,1} & u_{M,2} & \cdots & u_{M,M(N+1)} \end{bmatrix}, \quad \mathbf{F}^* = \begin{bmatrix} f_1(x_0) \\ f_2(x_0) \\ \vdots \\ f_M(x_{N-1}) \\ \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_M \end{bmatrix}.$$

Matrix equation (21) is a system of $M(N+1)\times M(N+1)$ nonlinear algebraic equations. After solving this system, the unknown coefficients vector $\mathbf{Y}$ can be uniquely determined and by putting it on the Eq. (10), approximate solution (4) and (5) in terms of FBPs can be obtained.

5. Error estimation and convergence analysis

In this section, it is tried to prove three theorems. In theorem 1, an upper bound is introduced for the error of function that is approximated by FBPs. In theorem 2 by considering the upper bound obtained in theorem 1, an upper bound is derived for the residual error function of Eqs. (1) and (2) when the unknown functions in this equations are approximated by FBPs. Theorem 3 offers an upper bound of absolute error for the collocation method in the certain case $q = 2$.

It is assumed that $X = L^2(I)$ is a Hilbert space and $\{B_i(x)\}_{i=0}^{N} \subset X$ as the set of FBPs of degree i. Let $X_N = span\{B_i(x)\,|i = 0, 1, \ldots, N\}$ and $\upsilon(x)$ be an arbitrary element in X. Given that X_N is a finite dimensional vector space and closed subspace of X, it can be said that the set of FBPs $\{B_i(x)\}_{i=0}^{N}$ constitute as a complete basis over the interval I. So, $\upsilon(x)$ has the unique best approximation out of X_N such as $\upsilon_N \in X_N$ and therefore, exist unique coefficients $\bar{\upsilon}_i$, $i = 0, 1, \ldots, N$ such that

$$\upsilon(x) \simeq \upsilon_N(x) = \sum_{i=0}^{N} \bar{\upsilon}_i\, B_i(x) = V^T B(x),$$

where V is a $(N+1)\times 1$ vector given by $V = [\bar{\upsilon}_0, \bar{\upsilon}_1, \ldots, \bar{\upsilon}_N]^T$.

Lemma 1. [43] Suppose that $\upsilon(x) \in C^{N+1}(I)$. In addition, let $p_N[\upsilon](x)$ is the interpolating polynomial of degree $\leq N$, that interpolates $\upsilon(x)$ on I at $N+1$ distinct points $x_i \in I,\ i = 0(1)N$. Then for each $x \in I$, there exists $\varepsilon_x \in (0, 1)$, which depends on x, such that

$$\upsilon(x) - p_N[\upsilon](x) = \frac{1}{(N+1)!}\upsilon^{(N+1)}(\varepsilon_x)\prod_{i=0}^{N}(x - x_i).$$

Theorem 1. Suppose that $\upsilon(x) \in C^{N+1}(I)$. If $V^T B(x)$ be the FBPs expansion of $\upsilon(x)$ in I, then the error bound is estimated as follows:

$$\left\|\upsilon(x) - V^T B(x)\right\|_\infty \leq \frac{\widehat{\sigma}}{4\,N^{N+1}(N+1)},$$

where $\widehat{\sigma}$ is non-negative constant, such that: $\max\limits_{\varepsilon_x\in I}\left|\upsilon^{(N+1)}(\varepsilon_x)\right| \leq \widehat{\sigma}$.

Proof. Since $V^T B(x)$ is the best approximation $\upsilon(x)$ out of X_N, we have

$$\left\|\upsilon(x) - V^T B(x)\right\|_\infty \leq \|\upsilon(x) - p_N[\upsilon](x)\|_\infty = \max_{x\in I}\,|\upsilon(x) - p_N[\upsilon](x)|, \tag{22}$$

where $p_N[\upsilon](x)$ interpolates $\upsilon(x)$ on I at points $x_i = \frac{i}{N}$, $i = 0\,(1)\,N$ with the following error bound

$$|\upsilon(x) - p_N[\upsilon](x)| \leq \frac{\widehat{\sigma}}{4\,N^{N+1}(N+1)}. \tag{23}$$

Therefore, (22) and (23) gives the desired result.

Theorem 2. Assume that $\{y_{i_1}(x)\}_{i_1=1}^{M} \in C^{N+1}(I)$ be the exact solution of Eqs. (1) and (2) and $R_N(x) = \sum\limits_{i_1=1}^{M} R_{i_1,N}(x)$ be the residual error function obtained by FBPs for solving this equation ($R_{i_1,N}(x)$ represent the residual error functions associated with the i_1-th equation of Eq. (1)). In addition, suppose the following assumptions hold.
(i) $e_{i_1,N}(x) = y_{i_1}(x) - y_{i_1,N}(x)$, $x \in I$, $i_1 = 1\,(1)\,M$ denote the error function of the FBPs approximation $y_{i_1,N}(x)$ to $y_{i_1}(x)$.
(ii) The nonlinear terms $(y_{i_1}(x))^q$ are satisfied the Lipschitz condition in such a manner that

$$\left\|(y_{i_1}(x))^q - (y_{i_1,N}(x))^q\right\|_\infty \leq L_{i_1}\left\|y_{i_1}(x) - y_{i_1,N}(x)\right\|_\infty = L_{i_1}\left\|e_{i_1,N}(x)\right\|_\infty, \quad x\in I,\ \ i_1 = 1\,(1)\,M,\ \ q = 2\,(1)\,\gamma,$$

where L_{i_1} are independent of $x,\ y_{i_1}(x),\ y_{i_1,N}(x)$.

(iii)

$$\left\|J_{i_1,i_2}(x)\right\|_\infty \le M\left(J_{i_1,i_2}\right),\quad \left\|J^{(1)}_{i_1,i_2}(x)\right\|_\infty \le M\left(J^{(1)}_{i_1,i_2}\right),\quad \left\|R^*_{i_1,i_2}(x)\right\|_\infty \le M\left(R^*_{i_1,i_2}\right),$$
$$\left\|\widehat{P}^{\,i_3}_{i_1,i_2}(x)\right\|_\infty = \frac{1}{|\mu_{i_3}|}\left\|P^{i_3}_{i_1,i_2}\left(\frac{x}{\mu_{i_3}}\right)\right\|_\infty \le M\left(\widehat{P}^{\,i_3}_{i_1,i_2}\right),\quad \left\|\widehat{P}^{*\,i_4}_{i_1,i_2}(x)\right\|_\infty = \frac{1}{|\sigma_{i_4}|}\left\|P^{*\,i_4}_{i_1,i_2}\left(\frac{x}{\sigma_{i_4}}\right)\right\|_\infty \le M\left(\widehat{P}^{*\,i_4}_{i_1,i_2}\right),$$
$$x\in I,\ \ i_1,\, i_2 = 1\,(1)\,M,\ \ i_3 = 1\,(1)\,\wp,\ \ i_4 = 1\,(1)\,\varpi.$$

Then the following inequality can be achieved for the function $R_N(x)$

$$\|R_N(x)\|_\infty \le \frac{1}{4\,N^{N+1}(N+1)}\ \left[C_1 + C_2 + C_3\right],$$

where

$$C_1 = \sum_{i_1=1}^{M}\sum_{i_2=1}^{M}\left(M\left(J_{i_1,i_2}\right) + M\left(J^{(1)}_{i_1,i_2}\right) + M\left(R^*_{i_1,i_2}\right)\right)\widehat{\sigma}_{i_2},\quad C_2 = \sum_{i_1=1}^{M}\sum_{i_3=1}^{\wp}\sum_{i_2=1}^{M}\left(M\left(\widehat{P}^{\,i_3}_{i_1,i_2}\right)\right)\widehat{\sigma}_{i_2},$$
$$C_3 = \sum_{i_1=1}^{M}\sum_{q=2}^{\gamma}\sum_{i_4=1}^{\varpi}\sum_{i_2=1}^{M} L_{i_2}\left(M\left(\widehat{P}^{*\,i_4}_{i_1,i_2}\right)\right)\widehat{\sigma}_{i_2},\quad \max_{x\in I}\left|y^{(N+1)}_{i_2}(x)\right| \le \widehat{\sigma}_{i_2}.$$

Proof. Integrating of both sides of Eq. (1) with respect to x from 0 to x and using the initial conditions (2), we have

$$\sum_{i_2=1}^{M} J_{i_1,i_2}(x)\,y_{i_2}(x) = \widehat{f}_{i_1}(x) + \sum_{i_2=1}^{M}\int_0^x\left(J^{(1)}_{i_1,i_2}(\tau) + R^*_{i_1,i_2}(\tau)\right)\,y_{i_2}(\tau)\,d\tau +$$
$$\sum_{i_3=1}^{\wp}\sum_{i_2=1}^{M}\int_0^x P^{i_3}_{i_1,i_2}(\tau)\,y_{i_2}(\mu_{i_3}\tau)\,d\tau + \sum_{q=2}^{\gamma}\sum_{i_4=1}^{\varpi}\sum_{i_2=1}^{M}\int_0^x P^{*\,i_4}_{i_1,i_2}(\tau)\left(y_{i_2}(\sigma_{i_4}\tau)\right)^q d\tau,\quad i_1 = 1\,(1)\,M,$$

where

$$\widehat{f}_{i_1}(x) = \int_0^x f_{i_1}(\tau)\,d\tau + \sum_{i_2=1}^{M}\lambda_{i_2} J_{i_1,i_2}(0).$$

Using the change of variables $\widehat{\tau} = \mu_{i_3}\tau$, $\widehat{\eta} = \sigma_{i_4}\tau$ and denoting $\widehat{P}^{\,i_3}_{i_1,i_2}(\widehat{\tau}) = \frac{1}{\mu_{i_3}}P^{i_3}_{i_1,i_2}(\frac{\widehat{\tau}}{\mu_{i_3}})$, $\widehat{P}^{*\,i_4}_{i_1,i_2}(\widehat{\eta}) = \frac{1}{\sigma_{i_4}}P^{*\,i_4}_{i_1,i_2}(\frac{\widehat{\eta}}{\sigma_{i_4}})$, the above equation can be transformed to the following equations system

$$\begin{aligned}&\sum_{i_2=1}^{M} J_{i_1,i_2}(x)\,y_{i_2}(x) = \widehat{f}_{i_1}(x) + \sum_{i_2=1}^{M}\int_0^x\left(J^{(1)}_{i_1,i_2}(\tau) + R^*_{i_1,i_2}(\tau)\right)\,y_{i_2}(\tau)\,d\tau + \\ &\sum_{i_3=1}^{\wp}\sum_{i_2=1}^{M}\int_0^{\mu_{i_3}x}\widehat{P}^{\,i_3}_{i_1,i_2}(\widehat{\tau})\,y_{i_2}(\widehat{\tau})\,d\widehat{\tau} + \sum_{q=2}^{\gamma}\sum_{i_4=1}^{\varpi}\sum_{i_2=1}^{M}\int_0^{\sigma_{i_4}x}\widehat{P}^{*\,i_4}_{i_1,i_2}(\widehat{\eta})\left(y_{i_2}(\widehat{\eta})\right)^q d\widehat{\eta},\quad i_1 = 1\,(1)\,M.\end{aligned}\tag{24}$$

Since $\{y_{i_1,N}(x)\}_{i_1=1}^{M}$ is the approximate solution of Eq. (1), hence satisfies in the Eq. (24) approximately. Consequently, the i_1-th equation of the above system can be represented as

$$\begin{aligned}&\sum_{i_2=1}^{M} J_{i_1,i_2}(x)\,y_{i_2,N}(x) = \widehat{f}_{i_1}(x) + \sum_{i_2=1}^{M}\int_0^x\left(J^{(1)}_{i_1,i_2}(\tau) + R^*_{i_1,i_2}(\tau)\right)\,y_{i_2,N}(\tau)\,d\tau + \\ &\sum_{i_3=1}^{\wp}\sum_{i_2=1}^{M}\int_0^{\mu_{i_3}x}\widehat{P}^{\,i_3}_{i_1,i_2}(\widehat{\tau})\,y_{i_2,N}(\widehat{\tau})\,d\widehat{\tau} + \sum_{q=2}^{\gamma}\sum_{i_4=1}^{\varpi}\sum_{i_2=1}^{M}\int_0^{\sigma_{i_4}x}\widehat{P}^{*\,i_4}_{i_1,i_2}(\widehat{\eta})\left(y_{i_2,N}(\widehat{\eta})\right)^q d\widehat{\eta} + R_{i_1,N}(x).\end{aligned}\tag{25}$$

Subtracting Eq. (25) from Eq. (24) and using assumption (i), yields:

$$-R_{i_1,N}(x) = \sum_{i_2=1}^{M} J_{i_1,i_2}(x)\,e_{i_2,N}(x) - \sum_{i_2=1}^{M}\int_0^x\left(J^{(1)}_{i_1,i_2}(\tau) + R^*_{i_1,i_2}(\tau)\right)\,e_{i_2,N}(\tau)\,d\tau -$$
$$\sum_{i_3=1}^{\wp}\sum_{i_2=1}^{M}\int_0^{\mu_{i_3}x}\widehat{P}^{\,i_3}_{i_1,i_2}(\widehat{\tau})\,e_{i_2,N}(\widehat{\tau})\,d\widehat{\tau} - \sum_{q=2}^{\gamma}\sum_{i_4=1}^{\varpi}\sum_{i_2=1}^{M}\int_0^{\sigma_{i_4}x}\widehat{P}^{*\,i_4}_{i_1,i_2}(\widehat{\eta})\ \left[\left(y_{i_2}(\widehat{\eta})\right)^q - \left(y_{i_2,N}(\widehat{\eta})\right)^q\right]d\widehat{\eta}.$$

Following Minkowski's inequality, using assumption (ii) and since $\mu_{i_3}x,\ \sigma_{i_4}x \in [0,\ 1)$, the following inequality can be derived

$$\left\|R_{i_1,N}(x)\right\|_\infty \le \sum_{i_2=1}^{M}\left\|J_{i_1,i_2}(x)\right\|_\infty\left\|e_{i_2,N}(x)\right\|_\infty + \sum_{i_2=1}^{M}\left(\left\|J^{(1)}_{i_1,i_2}(x)\right\|_\infty + \left\|R^*_{i_1,i_2}(x)\right\|_\infty\right)\left\|e_{i_2,N}(x)\right\|_\infty +$$
$$\sum_{i_3=1}^{\wp}\sum_{i_2=1}^{M}\left\|\widehat{P}^{\,i_3}_{i_1,i_2}(x)\right\|_\infty\left\|e_{i_2,N}(x)\right\|_\infty + \sum_{q=2}^{\gamma}\sum_{i_4=1}^{\varpi}\sum_{i_2=1}^{M} L_{i_2}\left\|\widehat{P}^{*\,i_4}_{i_1,i_2}(x)\right\|_\infty\left\|e_{i_2,N}(x)\right\|_\infty.$$

Since $\{y_{i_1,N}(x)\}_{i_1=1}^{M}$ is also the best approximations for $\{y_{i_1}(x)\}_{i_1=1}^{M}$, by using theorem 1 and assumption (iii), the following inequality yield

$$\left\|R_{i_1,N}(x)\right\|_\infty \le \frac{1}{4\,N^{N+1}(N+1)} \left[\sum_{i_2=1}^{M}\left(M\left(J_{i_1,i_2}\right)+M\left(J^{(1)}_{i_1,i_2}\right)+M\left(R^*_{i_1,i_2}\right)\right)\widehat{\sigma}_{i_2} + \sum_{i_3=1}^{\wp}\sum_{i_2=1}^{M}\left(M\left(\widehat{P}^{\,i_3}_{i_1,i_2}\right)\right)\widehat{\sigma}_{i_2} + \sum_{q=2}^{\gamma}\sum_{i_4=1}^{\varpi}\sum_{i_2=1}^{M} L_{i_2}\left(M\left(\widehat{P}^{*\,i_4}_{i_1,i_2}\right)\right)\widehat{\sigma}_{i_2}\right].$$

Or

$$\|R_N(x)\|_\infty \le \frac{1}{4\,N^{N+1}(N+1)}\,\left[C_1+C_2+C_3\right],$$

and this completes the proof.

Remark 1. Theorem 2 concluded that $\|R_N(x)\|_\infty = o\left(\frac{1}{N^{N+1}(N+1)}\right)$ and as $N \to +\infty$; the approximate solution (3) using the method proposed in previous section converges to the exact solution; that is, $\lim\limits_{N\to+\infty} y_{i_1,N}(x) = y_{i_1}(x)$, $i_1 = 1\,(1)\,M$.

Remark 2. Assume that $\{y_{i_1}(x)\}_{i_1=1}^{M}$ be the exact solution of Eqs. (1) and (2) and $p_N[y_{i_1}](x)$ be the interpolation polynomial corresponding to $y_{i_1}(x)$. If $\{y_{i_1}(x)\}_{i_1=1}^{M}$ is sufficiently smooth [42], according to lemma 1, $y_{i_1}(x)$ is written as $y_{i_1}(x) = p_N[y_{i_1}](x) + E_N[y_{i_1}](x)$ where

$$E_N[y_{i_1}](x) = \frac{1}{(N+1)!}{y_{i_1}}^{(N+1)}(\varepsilon_x)\prod_{i=0}^{N}(x-x_i),\quad \varepsilon_x\in(0,\,1),\quad i_1 = 1,\,\ldots,\,M.$$

Since $\{y_{i_1,N}(x)\}_{i_1=1}^{M}$ is the FBPs solution of Eq. (1), hence satisfies the Eq. (1) on the nodes. If $p_N[y_{i_1}](x)$ is approximated as $p_N[y_{i_1}](x) \simeq \sum\limits_{i=0}^{N}\tilde{y}_{i_1,i}\,B_i(x) = \mathbf{B}^T(x)\tilde{\mathbf{Y}}_{i_1}$, where $\tilde{y}_{i_1,i}$ is constant coefficient, then, $\{y_{i_1,N}(x)\}_{i_1=1}^{M}$ and $\{p_N[y_{i_1}](x)\}_{i_1=1}^{M}$ are the solutions of Eq. (1) which yield the nonlinear systems of $\mathbf{W}^*\,\mathbf{Y}-\mathbf{F}^* = \mathbf{0}$ and $\tilde{\tilde{\mathbf{W}}}\,\tilde{\mathbf{Y}}-\mathbf{F}^*-\Delta\mathbf{F}^* = \mathbf{0}$, respectively, where

$$\tilde{\tilde{\mathbf{W}}} = \begin{bmatrix} \tilde{w}_{1,1} & \tilde{w}_{1,2} & \cdots & \tilde{w}_{1,M(N+1)} \\ \tilde{w}_{2,1} & \tilde{w}_{2,2} & \cdots & \tilde{w}_{2,M(N+1)} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{w}_{MN,1} & \tilde{w}_{MN,2} & \cdots & \tilde{w}_{MN,M(N+1)} \\ u_{1,1} & u_{1,2} & \cdots & u_{1,M(N+1)} \\ u_{2,1} & u_{2,2} & \cdots & u_{2,M(N+1)} \\ \vdots & \vdots & \ddots & \vdots \\ u_{M,1} & u_{M,2} & \cdots & u_{M,M(N+1)} \end{bmatrix},\quad \tilde{\mathbf{Y}} = \begin{bmatrix} \tilde{\mathbf{Y}}_1 \\ \tilde{\mathbf{Y}}_2 \\ \vdots \\ \tilde{\mathbf{Y}}_M \end{bmatrix},\quad \Delta\mathbf{F}^* = \begin{bmatrix} g_1(x_0) \\ g_2(x_0) \\ \vdots \\ g_M(x_{N-1}) \\ -E_N[y_1](0) \\ -E_N[y_2](0) \\ \vdots \\ -E_N[y_M](0) \end{bmatrix},$$

$$\tilde{\mathbf{W}} = \left(\mathbf{J}\,\hat{\mathbf{T}}\,\hat{\mathbf{D}}_t - \mathbf{R}^*\,\hat{\mathbf{T}} - \sum_{i_3=1}^{\wp}\mathbf{P}_{i_3}\,\hat{\mathbf{T}}\,\hat{\mathbf{Z}}_{\mu_{i_3}} - \sum_{i_4=1}^{\varpi}\left(\mathbf{P}^*_{i_4}\,\hat{\mathbf{T}}\,\hat{\mathbf{Z}}_{\sigma_{i_4}}\,\hat{\hat{\mathbf{A}}}\,\tilde{\tilde{\mathbf{Y}}}\,\hat{\mathbf{T}}\,\hat{\mathbf{Z}}_{\sigma_{i_4}} + 2\,\mathbf{PE}_{i_4}\,\hat{\mathbf{T}}\,\hat{\mathbf{Z}}_{\sigma_{i_4}}\right)\right)\hat{\mathbf{A}},$$

$$\tilde{\tilde{\mathbf{Y}}} = \mathrm{Diag}\left(\tilde{\mathbf{Y}},\,\tilde{\mathbf{Y}},\,\ldots,\,\tilde{\mathbf{Y}}\right),\quad \tilde{\mathbf{Y}} = \mathrm{Diag}\left(\tilde{\mathbf{Y}}_1,\,\tilde{\mathbf{Y}}_2,\,\ldots,\,\tilde{\mathbf{Y}}_M\right),$$

and

$$\mathbf{PE}_{i_4} = \text{Diag}\left(\mathbf{PE}_{i_4}(x_0), \mathbf{PE}_{i_4}(x_1), \ldots, \mathbf{PE}_{i_4}(x_N)\right),$$

$$\mathbf{PE}_{i_4}(x) = \begin{bmatrix} P^{*\,i_4}_{1,1}(x)\,E_N[y_1](\sigma_{i_4}x) & P^{*\,i_4}_{1,2}(x)\,E_N[y_2](\sigma_{i_4}x) & \ldots & P^{*\,i_4}_{1,M}(x)\,E_N[y_M](\sigma_{i_4}x) \\ P^{*\,i_4}_{2,1}(x)\,E_N[y_1](\sigma_{i_4}x) & P^{*\,i_4}_{2,2}(x)E_N[y_2](\sigma_{i_4}x) & \ldots & P^{*\,i_4}_{2,M}(x)\,E_N[y_M](\sigma_{i_4}x) \\ \vdots & \vdots & \ddots & \vdots \\ P^{*\,i_4}_{M,1}(x)\,E_N[y_1](\sigma_{i_4}x) & P^{*\,i_4}_{M,2}(x)E_N[y_2](\sigma_{i_4}x) & \ldots & P^{*\,i_4}_{M,M}(x)\,E_N[y_M](\sigma_{i_4}x) \end{bmatrix},$$

$$g_{i_1}(x) = -\sum_{i_2=1}^{M} J_{i_1,i_2}(x)\,E_N^{(1)}[y_{i_2}](x) + \sum_{i_2=1}^{M} R^{*}_{i_1,i_2}(x)\,E_N[y_{i_2}](x) + \sum_{i_3=1}^{\wp}\sum_{i_2=1}^{M} P^{i_3}_{i_1,i_2}(x)\,E_N[y_{i_2}](\mu_{i_3}x) + \sum_{i_4=1}^{\varpi}\sum_{i_2=1}^{M} P^{*\,i_4}_{i_1,i_2}(x)\left(E_N[y_{i_2}](\sigma_{i_4}x)\right)^2 \quad i_1 = 1, \ldots, M.$$

According to the above remark, now we can prove the following theorem regarding the error due to the collocation procedure.

Theorem 3. Let $\mathbf{y}(x) = [y_1, y_2, \ldots, y_M]^T$ and $\mathbf{y}_N(x) = [y_{1,N}, y_{2,N}, \ldots, y_{M,N}]^T$ are the exact and approximate solutions of Eq. (1) respectively and $y_{i_1}(x) \in C^{N+1}(I)$, then

$$\|\mathbf{y}(x) - \mathbf{y}_N(x)\|_\infty \leq \max_{\substack{1\leq i_1\leq M \\ x\in I}} \left\{ \frac{1}{(N+1)!} \left| {y_{i_1}}^{(N+1)}(\varepsilon_x) \prod_{i=0}^{N} (x - x_i) \right| + \left\|\mathbf{B}^T(x)\right\|_\infty \left\|\mathbf{Y}_{i_1} - \tilde{\mathbf{Y}}_{i_1}\right\|_\infty \right\}.$$

Proof. Since $y_{i_1}(x) \in C^{N+1}(I)$, using triangle inequality and remark 2, for each element of $\mathbf{y}(x) - \mathbf{y}_N(x)$, we can obtain

$$\left|y_{i_1}(x) - y_{i_1,N}(x)\right| \leq \left|y_{i_1}(x) - p_N[y_{i_1}](x)\right| + \left|y_{i_1,N}(x) - p_N[y_{i_1}](x)\right| = \frac{1}{(N+1)!}\left| {y_{i_1}}^{(N+1)}(\varepsilon_x) \prod_{i=0}^{N}(x - x_i)\right| + \left|y_{i_1,N}(x) - p_N[y_{i_1}](x)\right|. \tag{26}$$

According to Eq. (8) and remark 2, we get

$$\left|y_{i_1,N}(x) - p_N[y_{i_1}](x)\right| = \left|\mathbf{B}^T(x)\mathbf{Y}_{i_1} - \mathbf{B}^T(x)\tilde{\mathbf{Y}}_{i_1}\right| = \left|\mathbf{B}^T(x)\left(\mathbf{Y}_{i_1} - \tilde{\mathbf{Y}}_{i_1}\right)\right| \leq \left\|\mathbf{B}^T(x)\right\|_\infty \left\|\mathbf{Y}_{i_1} - \tilde{\mathbf{Y}}_{i_1}\right\|_\infty. \tag{27}$$

Using Eqs. (26) and (27) finally we obtain the desired result.

Table 1. Comparison between absolute error functions obtained by proposed approach and method in [3] for Example 1.

Nodes x	Method in [3]	Present approach ($N = 9$)
0.01	$6.19E-04$	$2.05E-11$
0.04	$6.40E-04$	$9.83E-12$
0.09	$6.77E-04$	$4.45E-11$
0.16	$7.29E-04$	$1.41E-11$
0.25	$7.93E-04$	$6.54E-11$
0.36	$8.59E-04$	$1.19E-10$
0.49	$9.04E-04$	$1.61E-10$
0.64	$8.85E-04$	$3.37E-10$
0.81	$7.35E-04$	$3.81E-10$

6. Numerical examples

In this section, three test problems were selected from references [3, 37] and have been resolved with the process described in section 4. In all examples, the Matlab function fsolve has been used to solve the existing system, which

Table 2. Absolute error functions of Example 1 obtained by proposed approach for $N = 3, 5, 8, 9$.

Nodes x	$\lvert e_3(x)\rvert$	$\lvert e_5(x)\rvert$	$\lvert e_8(x)\rvert$	$\lvert e_9(x)\rvert$
0.0	$4.71E-14$	$4.84E-13$	$7.23E-12$	$2.38E-11$
0.2	$8.16E-03$	$4.29E-05$	$3.29E-09$	$2.18E-11$
0.4	$1.22E-02$	$3.10E-05$	$3.75E-09$	$1.25E-10$
0.6	$2.85E-03$	$4.45E-05$	$3.72E-09$	$2.95E-10$
0.8	$6.22E-03$	$3.59E-06$	$2.71E-09$	$3.51E-10$
1.0	$9.31E-02$	$6.99E-04$	$1.08E-07$	$9.01E-09$

uses Newton's as the default method and application of this function has been instrumental in obtaining an accurate approximate solution of the system. This happended despite starting with a zero initial approximation.
Calculated values are the values of absolute error functions, L_2 errors and maximum absolute error functions and are defined as $\left|e_{i_1,N}(x)\right|$, $\left\|e_{i_1,N}(x)\right\|_2 = \left(\int_0^1 (e_{i_1,N}(x))^2 dx\right)^{\frac{1}{2}}$, $\left\|e_{i_1,N}(x)\right\|_\infty = \max_{x\in I} \left|e_{i_1,N}(x)\right|$, $i_1 = 1\,(1)\,M$. In case $M = 1$ functions $e_{i_1,N}(x)$ are denoted by $e_N(x)$.
Numerical results are tabulated in Tables 1-7. As the Tables show, by increasing N, the value of the above errors dramatically reduced. In addition, some of the obtained results have been compared with the results of available methods. The results confirm that the FBPs have higher accuracy in comparison with the methods [3, 37].

Table 3. Comparison between maximum absolute error functions of Example 2 obtained by proposed approach and method in [37] for $N = 1, 2, 3$ in the interval I.

N	$\left\|e_{1,N}(x)\right\|_\infty$		$\left\|e_{2,N}(x)\right\|_\infty$	
	Method in [37]	Present approach	Method in [37]	Present approach
1	$4.19E-01$	$1.99E-01$	$8.77E-01$	$1.59E-01$
2	$8.93E-02$	$2.98E-02$	$1.97E-01$	$5.30E-02$
3	$1.96E-02$	$1.06E-02$	$1.02E-02$	$1.41E-03$

Table 4. Absolute error functions of Example 2 obtained by proposed approach for $N = 4, 6, 9$.

Nodes x	$\lvert e_{1,4}(x)\rvert$	$\lvert e_{1,6}(x)\rvert$	$\lvert e_{1,9}(x)\rvert$
0.2	$7.50E-05$	$1.30E-07$	$1.22E-11$
0.4	$3.02E-05$	$7.26E-08$	$9.91E-12$
0.6	$2.71E-05$	$4.04E-08$	$7.20E-12$
0.8	$5.41E-05$	$1.04E-07$	$6.56E-12$
1.0	$1.12E-03$	$2.35E-06$	$7.58E-10$
Nodes x	$\lvert e_{2,4}(x)\rvert$	$\lvert e_{2,6}(x)\rvert$	$\lvert e_{2,9}(x)\rvert$
0.2	$3.12E-05$	$6.88E-08$	$3.55E-12$
0.4	$4.91E-05$	$1.19E-07$	$1.04E-11$
0.6	$8.39E-05$	$1.60E-07$	$1.59E-11$
0.8	$1.16E-04$	$2.16E-07$	$2.06E-11$
1.0	$2.00E-04$	$7.61E-07$	$1.47E-11$

Example 1. Consider the first order nonlinear pantograph delay differential equation [3]:

$$y^{(1)}(x) = y(x) + x\,y^2(0.5\,x) + f(x), \quad x \in I,$$

with the initial condition $y(0) = 0$ and the exact solution $y(x) = 9\,x\,(e^{x-1} - 1)$. Here, $f(x) = 9\,(e^{x-1} - 1) + 9\,x\,e^{x-1} - 9\,x\,(e^{x-1} - 1) - \frac{81}{4}\,x^3(e^{\frac{x}{2}-1} - 1)^2$. For numerical results see Tables 1, 2.

Table 5. Comparison between maximum absolute error functions of Example 3 obtained by proposed approach and method in [37] for $N = 1, 2, 3$ in the interval I.

N	$\|e_{1,N}(x)\|_\infty$ Method in [37]	$\|e_{1,N}(x)\|_\infty$ Present approach	$\|e_{2,N}(x)\|_\infty$ Method in [37]	$\|e_{2,N}(x)\|_\infty$ Present approach	$\|e_{3,N}(x)\|_\infty$ Method in [37]	$\|e_{3,N}(x)\|_\infty$ Present approach
1	$5.92E-01$	$4.60E-01$	$1.53E-01$	$4.60E-01$	$3.59E-01$	$1.59E-01$
2	$1.93E-01$	$2.38E-02$	$1.08E-01$	$1.03E-01$	$2.33E-01$	$1.40E-02$
3	$6.77E-02$	$2.80E-03$	$1.89E-02$	$9.69E-03$	$3.39E-02$	$1.30E-03$

Example 2. Consider the two-dimensional system of nonlinear multi pantograph delay differential equations [37]:

$$\begin{cases} y_1^{(1)}(x) = -y_1(x) - 2\,e^{-\frac{3}{4}x}\cos\left(\frac{1}{2}\,x\right)\,\sin\left(\frac{1}{4}\,x\right) y_1(0.25\,x) - e^{-x}\cos\left(\frac{1}{2}\,x\right) y_2(0.5\,x), \\ y_2^{(1)}(x) = e^x y_1^2(0.5\,x) - y_2^2(0.5\,x), \end{cases} \quad x \in I,$$

with the initial conditions $y_1(0) = 1$, $y_2(0) = 0$ and the exact solution $y_1(x) = e^{-x}\cos(x)$, $y_2(x) = \sin(x)$. For numerical results see Tables 3, 4 and 7.

Table 6. Absolute error functions of Example 3 obtained by proposed approach for $N = 4, 6, 8$.

Nodes x	$\|e_{1,4}(x)\|$	$\|e_{1,6}(x)\|$	$\|e_{1,8}(x)\|$
0.2	$2.38E-05$	$7.78E-08$	$1.56E-10$
0.4	$6.28E-05$	$2.15E-07$	$4.04E-10$
0.6	$1.17E-04$	$3.42E-07$	$6.19E-10$
0.8	$1.68E-04$	$4.63E-07$	$8.36E-10$
1.0	$2.77E-05$	$4.70E-08$	$4.40E-11$
Nodes x	$\|e_{2,4}(x)\|$	$\|e_{2,6}(x)\|$	$\|e_{2,8}(x)\|$
0.2	$1.11E-04$	$3.10E-07$	$4.74E-10$
0.4	$7.13E-05$	$2.35E-07$	$4.76E-10$
0.6	$6.93E-05$	$2.00E-07$	$4.55E-10$
0.8	$1.13E-04$	$3.81E-07$	$4.07E-10$
1.0	$1.28E-03$	$6.25E-06$	$1.60E-08$
Nodes x	$\|e_{3,4}(x)\|$	$\|e_{3,6}(x)\|$	$\|e_{3,8}(x)\|$
0.2	$3.70E-05$	$9.54E-08$	$1.43E-10$
0.4	$4.18E-05$	$1.05E-07$	$1.81E-10$
0.6	$3.54E-05$	$9.70E-08$	$1.68E-10$
0.8	$3.88E-05$	$8.13E-08$	$9.89E-11$
1.0	$2.06E-04$	$7.03E-07$	$1.37E-09$

Example 3. Consider the three-dimensional system of nonlinear pantograph delay differential equations [37]:

$$\begin{cases} y_1^{(1)}(x) = y_3(x) + 2\,y_2(0.5\,x) + f_1(x), \\ y_2^{(1)}(x) = -2\,y_3^2(0.5\,x) + f_2(x), \\ y_3^{(1)}(x) = -y_1(x) + y_2(x) + f_3(x), \end{cases} \quad x \in I,$$

with the initial conditions $y_1(0) = -1$, $y_2(0) = 0$, $y_3(0) = 0$ and the exact solution $y_1(x) = -\cos(x)$, $y_2(x) = x\cos(x)$, $y_3(x) = \sin(x)$. Here, $f_1(x) = -x\cos\left(\frac{1}{2}\,x\right)$, $f_2(x) = 1 - x\sin(x)$, and $f_3(x) = -x\cos(x)$. For numerical results see Tables 5, 6 and 7.

7. Conclusions

FBPs were introduced to numerically solve Eq. (1) under the initial conditions (2). The noteworthy feature of the proposed method in section 4 is that due to the matrix form of solution algorithm, it is very easy to write a computer program. On the other hand, because the most elements of appeared matrices in the solution method are zeros, therefore, the calculation time is very short. A reasonable upper bound was presented for the residual error function and thus it was proved that the method is convergent and its order of convergence is $o\left(\frac{1}{N^{N+1}(N+1)}\right)$. The method was tested on three numerical examples and as can be seen in Tables 1-7, for small quantities N ($8 \leq N \leq 11$), an approximation with high accuracy ($10^{-12} \leq \left|e_{i_1,N}(x)\right| \leq 10^{-8}$) can be obtained. The presented method can be easily generalized to solve system of Volterra-Fredholm integro-differential-difference equations and system of partial differential equations but some reformations are needed.

Table 7. L_2 errors obtained by proposed approach for $N = 1\,(1)\,10$.

N	Example 2		Example 3		
	$\|e_{1,N}(x)\|_2$	$\|e_{2,N}(x)\|_2$	$\|e_{1,N}(x)\|_2$	$\|e_{2,N}(x)\|_2$	$\|e_{3,N}(x)\|_2$
1	$8.05E-02$	$6.06E-02$	$2.11E-01$	$1.77E-01$	$6.06E-02$
2	$9.90E-03$	$1.59E-02$	$1.58E-02$	$3.13E-02$	$9.70E-03$
3	$2.50E-03$	$8.00E-04$	$6.41E-04$	$2.20E-03$	$4.10E-04$
4	$2.16E-04$	$7.26E-05$	$1.00E-04$	$2.48E-04$	$4.86E-05$
5	$8.01E-06$	$2.74E-06$	$2.67E-06$	$1.12E-05$	$1.53E-06$
6	$3.61E-07$	$1.64E-07$	$2.99E-07$	$9.47E-07$	$1.29E-07$
7	$6.43E-08$	$1.69E-08$	$6.01E-09$	$2.59E-08$	$2.69E-09$
8	$3.70E-09$	$8.34E-10$	$5.62E-10$	$2.04E-09$	$2.14E-10$
9	$9.05E-11$	$1.39E-11$	$1.10E-09$	$6.15E-09$	$8.90E-10$
10	$5.44E-12$	$5.58E-11$	$1.78E-11$	$3.79E-11$	$1.16E-11$

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