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Generalization of Dunkerley’s equation for the undamped linear positive semidefinite system

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Abstract

The Dunkerley's equation can be generalized for the undamped linear positive semidefinite system (p s-d system) based on a new concise method for making the system's matrices nonsingular, which was advanced by the authors [9]. It was found that this kind of system has similar characteristic as the positive definite system—its fundamental frequency can be estimated with its basic subsystems by a formula which is similar to Dunkerley's equation. For the undamped linear p s-d system, in which masses are connected in series with springs, the reciprocal of its fundamental frequency squared is approximately equal to the sum of the reciprocals of the natural frequencies squared as the system is divided into two groups of masses rigidized according to their connection relationship of springs.

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1. Introduction

Dunkerley’s equation is one of the classical methods, which results in a lower bound to the fundamental frequency of linear Multi-Degree-of-Freedom vibration system. This method is of great value and importance. According to this method, a series of equations were put forward [1-8], for example, lablonskii et al. in their book [2] introduced several similar equations, which were more precision, can result the upper
and lower bound of fundamental frequency, can result the second order of fundamental frequency. Sheng Shanding et al. [6] generalized Dunkerley’s equation to the continuous system, Yao Zuoqiu [7] deduced two new and simple approximate formulas on the natural frequencies, Yan Shichao [8] generalized Dunkerley-Southwell’s formula for structure-foundation system.

The Dunkerley’s equation and its successors can not be used directly in positive semidefinite system (p s-d system), since that its dynamical matrix isn’t existed. The authors [9] found a new concise method for making the mass matrix and stiffness matrix of p s-d system nonsingular, which provides a point of vantage to generalize the Dunkerley’s equation for p s-d system. Based on this method, the corresponding method and equation were put forward in this paper.

2. Dunkerley’s equation

There is a undamped linear Multi-Degree-of-Freedom (multi-DOF) vibration system, which equation of motion is \[ [m]\ddot{x} + [k]x = 0, \] where \([m]\) is mass matrix, \([k]\) is stiffness matrix. The equation of motion can be written as \[ [D]\ddot{x} + [x] = 0, \] in which \([D] = [k]^{-1}[m]\) is dynamical matrix. Dunkerley’s estimate of fundamental frequency is then made from the equation

\[
\frac{1}{\omega_i^2} \approx \text{Trace}[D],
\]

or

\[
\frac{1}{\omega_i^2} \approx \frac{1}{\omega_{11}^2} + \frac{1}{\omega_{22}^2} + \cdots + \frac{1}{\omega_{nn}^2}.
\]

where \(\omega_i\) = fundamental frequency of the system,
\(\omega_{ii}\) = fundamental frequency in the absence of all other masses as the system is treated as a one DOF system.

3. The method of generalization Dunkerley’s equation for p s-d system

As a matter of convenience, a 3-DOF system is discussed as an example, which is shown in figure 1.

![3-DOF vibration system](image_url)
The equation of motion of this system is

\[
\begin{bmatrix}
  m_1 & 0 & 0 \\
  0 & m_2 & 0 \\
  0 & 0 & m_3
\end{bmatrix}
\begin{bmatrix}
  \ddot{x}_1 \\
  \ddot{x}_2 \\
  \ddot{x}_3
\end{bmatrix}
+ \begin{bmatrix}
  k_{12} & -k_{12} & 0 \\
  -k_{12} & k_{12} + k_{23} & -k_{23} \\
  0 & -k_{23} & k_{23}
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix}
= \begin{bmatrix}
  0 \\
  0 \\
  0
\end{bmatrix}.
\]

(3)

In order to use Dunkerley’s equation to estimate its fundamental frequency, the stiffness matrix must be nonsingular. Using the traditional method, the equation (3) can be changed to

\[
\begin{align*}
\frac{1}{m_3} & \begin{bmatrix}
  m_1m_3 + m_1^2 & m_1m_2 & m_1m_2 \\
  m_1m_2 & m_2m_3 + m_2^2 & m_2m_2 \\
  m_1m_2 & m_2m_2 & m_3m_3 + m_3^2
\end{bmatrix}
\begin{bmatrix}
  \ddot{x}_1 \\
  \ddot{x}_2 \\
  \ddot{x}_3
\end{bmatrix} \\
+ & \begin{bmatrix}
  k_{12}m_3^2 + k_{23}m_1^2 & -k_{12}m_3^2 + k_{23}m_1(m_2 + m_3) \\
  -k_{12}m_3^2 + k_{23}m_1(m_2 + m_3) & (k_{12} + k_{23})m_2^2 + k_{23}m_1(m_2 + 2m_3)
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0
\end{bmatrix}.
\end{align*}
\]

Obviously this equation is too complex to transform further, and a more concise method should be adopted. Using the new method deduced by the authors [9], a more simple equation can be gained, in which mass matrix and stiffness matrix have been nonsingular:

\[
\begin{bmatrix}
  m_1 & 0 \\
  0 & m_2
\end{bmatrix}
\begin{bmatrix}
  \ddot{x}_1 \\
  \ddot{x}_2
\end{bmatrix}
+ \begin{bmatrix}
  k_{12} & -k_{12} \\
  k_{12}m_3 - k_{12} & k_{12} + k_{23} + k_{23}m_3/m_3
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0
\end{bmatrix}.
\]

Then, the dynamical matrix is

\[
D = \frac{1}{m_1 + m_2 + m_3}
\begin{bmatrix}
  m_1(m_2 + m_3) + m_1m_3 & m_1m_2 \\
  m_1m_2 & m_2m_3 + m_2^2
\end{bmatrix}
\begin{bmatrix}
  k_{12} \\
  k_{23}
\end{bmatrix}
\begin{bmatrix}
  m_1 \\
  m_2
\end{bmatrix}.
\]

Substituting into equation (1), we obtain

\[
\frac{1}{\omega_i^2} \approx \frac{1}{m_1 + m_2 + m_3}
\begin{bmatrix}
  m_1(m_2 + m_3) \\
  k_{12}
\end{bmatrix}
+ \frac{m_1(m_2 + m_3)}{k_{23}}.
\]

(4)
This example shows that the following steps should be adopted in order to use Dunkerley’s equation for p s-d system. The first step is to make the stiffness matrix nonsingular by the new concise method deduced by the authors. The second step is to obtain the dynamical matrix, and the final step is to substitute into equation (1), obtain the lower bound to the fundamental frequency.

4. A new equation for estimating fundamental frequency of p s-d system

For the sake of finding the regularity, expand equation (4). The first term is \( \frac{1}{k_{12}} \frac{m_1(m_2 + m_3)}{m_1 + m_2 + m_3} \). Its reciprocal equals to \( k_{12} \left( \frac{1}{m_1} + \frac{1}{m_2 + m_3} \right) \), equals to the square of fundamental frequency of a imaginary system in absence of the spring \( k_{23} \), viz. in this imaginary system the mass \( m_2 \) and mass \( m_3 \) are rigidized as one mass. And the second term of equation (4) has the analogical meaning, it is equal to the square of reciprocal of fundamental frequency of a imaginary system in absence of the spring \( k_{12} \). Introducing two new labels \( \omega_{1,23} \) and \( \omega_{12,3} \) (\( \omega_{1,23} = \sqrt{k_{12} \left( \frac{1}{m_1} + \frac{1}{m_2 + m_3} \right)} \), \( \omega_{12,3} = \sqrt{k_{12} \left( \frac{1}{m_1} + \frac{1}{m_2 + m_3} \right)} \)), the equation (4) can be written as:

\[
\frac{1}{\omega_i^2} \approx \frac{1}{\omega_{1,23}^2} + \frac{1}{\omega_{12,3}^2}. \tag{5}
\]

In order to gain a general equation, a 5-DOF system is analyzed, which is shown in figure 2.

![Fig. 2. a 5-DOF vibration system](image)

Using the same method, we obtain

\[
\frac{1}{\omega_i^2} \approx \frac{1}{\sum_{i=1}^{5} m_i} \left[ \frac{\sum_{i=1}^{4} m_i}{k_{12}} + \frac{\sum_{i=2}^{5} m_i}{k_{23}} + \frac{\sum_{i=1}^{5} m_i}{k_{34}} + \frac{\sum_{i=4}^{5} m_i}{k_{45}} \right].
\]
Adopting the similar label, this equation can be written as

\[
\frac{1}{\omega^2} \approx \frac{1}{\omega_{1,23,45}^2} + \frac{1}{\omega_{12,345}^2} + \frac{1}{\omega_{123,45}^2} + \frac{1}{\omega_{1234,5}^2}.
\]

Furthermore, other similar systems are studied, including 4-DOF system and 6-DOF system, in which all the masses are connected in series with springs, and the similar equation for estimating fundamental frequency were obtained. Hereby, it is reasonable to conclude that for the undamped linear p s-d system, in which masses are connected in series with springs, the reciprocal of its fundamental frequency squared is approximately equal to the sum of the reciprocals of the natural frequencies squared in absence of all other springs as the system is divided into two groups of masses rigidized according to their connection relationship of springs. The general equation for estimating fundamental frequency of linear p s-d system is as follow:

\[
\frac{1}{\omega^2} \approx \frac{1}{\omega_{1,2,3-4}^2} + \frac{1}{\omega_{12,3-4}^2} + \cdots + \frac{1}{\omega_{12\cdots i\cdots j\cdots l\cdots m}^2} + \cdots + \frac{1}{\omega_{12\cdots n-1,n}^2}.
\]

where \(\omega_{1,2,3-4} = \text{fundamental frequency of a imaginary system, in which only spring } k_{ij} \text{ is held, viz. in the absence of all other springs, and the system is divided into two groups of masses, one group includes the masses } m_i, m_{i+1}, \cdots, m_j \) and the other group includes the masses \(m_{i+1}, m_{i+2}, \cdots, m_n\).

Comparing the equation (2) and equation (6), we can find some regularity, Dunkerley’s equation shows that the fundamental frequency of a positive definite system can be estimated with its basic subsystems (one-DOF positive definite system), in which all the other masses is absent. The equation (6) shows that the fundamental frequency of a positive semidefinite system also can be estimated with its basic subsystems (two mass p s-d system), in which all the other springs is absent. Therefore, not only positive definite system, but positive semidefinite system has a similar characteristic viz. their fundamental frequency can be estimated by basic subsystems.

5. Conclusions

The estimate method of fundamental frequency of undamped linear p s-d system was studied in this paper, and a general estimate equation was obtained, which is similar to Dunkerley’s equation. The following conclusions can be made:

(1) Dunkerley’s equation can be used in linear p s-d system based on the method deduced by the authors.
(2) The fundamental frequency of linear p s-d system can be estimated by equation (6), which is similar to Dunkerley’s equation.
(3) Positive definite system and positive semidefinite system has a similar characteristic viz. their fundamental frequency can be estimated by their basic subsystems.

References:


