Finite Dimensional Hilbert Spaces are Complete for Dagger Compact Closed Categories
(Extended Abstract)

Peter Selinger[1]

Department of Mathematics and Statistics
Dalhousie University, Halifax, Nova Scotia, Canada

Abstract

We show that an equation follows from the axioms of dagger compact closed categories if and only if it holds in finite dimensional Hilbert spaces.

Keywords: Dagger compact closed categories, Hilbert spaces, completeness.

1 Introduction

Hasegawa, Hofmann, and Plotkin recently showed that the category of finite dimensional vector spaces over any fixed field k of characteristic 0 is *complete* for traced symmetric monoidal categories [2]. What this means is that an equation holds in all traced symmetric monoidal categories if and only if it holds in finite dimensional vector spaces. Via Joyal, Street, and Verity's "Int"-construction [3], it is a direct corollary that finite dimensional vector spaces are also complete for compact closed categories. The present paper makes two contributions: (1) we simplify the proof of Hasegawa, Hofmann, and Plotkin's result, and (2) we extend it to show that finite dimensional *Hilbert* spaces are complete for *dagger* traced symmetric monoidal categories (and hence for dagger compact closed categories).

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2   Statement of the main result

For a definition of dagger compact closed categories, their term language, and their
graphical language, see [1,4]. We also use the concept of a *dagger traced monoidal
category*, which is a dagger symmetric monoidal category [4] with a trace operation
[3] satisfying $\mathrm{Tr}_{U,V}^{X}(f)^{\dagger} = \mathrm{Tr}_{V,U}^{X}(f^{\dagger})$. We note that every dagger compact closed
category is also dagger traced monoidal; conversely, by Joyal, Street, and Verity's
"Int" construction, every dagger traced monoidal category can be fully embedded
in a dagger compact closed category.

   We will make use of the soundness and completeness of the graphical represen-
tation, specifically of the following result:

Theorem 2.1 ([4]) *A well-typed equation between morphisms in the language of
dagger compact closed categories follows from the axioms of dagger compact closed
categories if and only if it holds, up to graph isomorphism, in the graphical language.*

   An analogous result also holds for dagger traced monoidal categories.
   The goal of this paper is to prove the following:

Theorem 2.2 *Let $M, N : A \to B$ be two terms in the language of dagger compact
closed categories. Suppose that $[\![M]\!] = [\![N]\!]$ for every possible interpretation (of
object variables as spaces and morphism variables as linear maps) in finite dimen-
sional Hilbert spaces. Then $M = N$ holds in the graphical language (and therefore,
holds in all dagger compact closed categories).*

3   Reductions

Before attempting to prove Theorem 2.2, we reduce the statement to something
simpler. By arguments analogous to those of Hasegawa, Hofmann, and Plotkin
[2], it suffices without loss of generality to consider terms M, N that satisfy some
additional conditions. The additional conditions are:

- We may assume that $M, N : I \to I$, i.e., that both the domain and codomain of
  M and N are the tensor unit. Such terms are called *closed*. The restriction to
  closed terms is without loss of generality, because given general $M, N : A \to B$,
  we can extend the language with two new morphism variables $f : I \to A$ and
  $g : B \to I$, and apply the theorem to the terms $M' = g \circ M \circ f$ and $N' = g \circ N \circ f$.
  Since g, f are new symbols, $g \circ M \circ f = g \circ N \circ f$ in the graphical language implies
  that $M = N$ in the graphical language.

- It suffices to consider terms M, N in the language of *dagger traced monoidal*
  categories. Namely, by Joyal, Street, and Verity's "Int"-construction [3], ev-
  ery statement about dagger compact closed categories can be translated to an
  equivalent statement about dagger traced monoidal categories. This is done by
  eliminating occurrences of the $*$-operation: one replaces every morphism variable
  such as $f : A^* \otimes B \otimes C^* \to D^* \otimes E$ by an equivalent new morphism variable such
  as $f' : B \otimes D \to A \otimes C \otimes E$ that does not use the $*$-operation.

- It suffices to consider terms whose graphical representation does not contain any "trivial cycles". Trivial cycles are connected components of a diagram that do not contain any morphism variables, such as the ones obtained from the trace of an identity morphism. The restriction is without loss of generality because if M, N have different numbers or types of trivial cycles, they can be easily separated in Hilbert spaces [2]. We say that a diagram is *simple* if it contains no trivial cycles.

4 Informal outline of the result

The formal statement and proof of Theorem 2.2 requires a fair amount of notation, and will be given elsewhere. Nevertheless, the main idea is simple, and we informally illustrate it here.

4.1 Signatures, diagrams, and interpretations

We assume given a set of *object variables*, denoted A, B etc., and a set of *morphism variables*, denoted f, g etc. A *sort* A is a finite sequence of object variables. We usually write $A_1 \otimes \ldots \otimes A_n$ for an n-element sequence, and I for the empty sequence. We assume that each morphism variable f is assigned two fixed sorts, called its *domain* A and *codomain* B respectively, and we write $f : A \to B$. We further require a fixpoint-free involution $(-)^\dagger$ on the set of morphism variables, such that $f^\dagger : B \to A$ when $f : A \to B$.

The collection of object variables and morphism variables, together with the domain and codomain information and the dagger operation is called a *signature* Σ of dagger monoidal categories.

Graphically, we represent a morphism variable $f : A_1 \otimes \ldots \otimes A_n \to B_1 \otimes \ldots \otimes B_m$ as a box
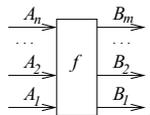
The wires on the left are called the *inputs* of f, and the wires on the right are called its *outputs*. Note that each box is labeled by a morphism variable, and each wire is labeled by an object variable.

A (closed simple dagger symmetric traced monoidal) *diagram* over a signature Σ consists of zero or more boxes of the above type, all of whose wires have been connected in pairs, such that each connection is between the output wire of some box and the input wire of some (possibly the same, possible another) box. Here is an example of a diagram N over the signature given by $f : B \to A \otimes A$, $g : A \otimes B \to$

$B \otimes A$.
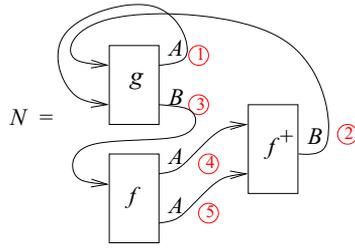
In the illustration, we have numbered the wires 1 to 5 to aid the exposition below; note that this numbering is not formally part of the diagram.

An *interpretation* of a signature in finite-dimensional Hilbert spaces consists of the following data: for each object variable A, a chosen finite-dimensional Hilbert space $[\![A]\!]$, and for each morphism variable $f : A_1 \otimes \ldots \otimes A_n \to B_1 \otimes \ldots \otimes B_m$, a chosen linear map $[\![f]\!] : [\![A_1]\!] \otimes \ldots \otimes [\![A_n]\!] \to [\![B_1]\!] \otimes \ldots \otimes [\![B_m]\!]$, such that $[\![f^\dagger]\!] = [\![f]\!]^\dagger$.

The *denotation* of a diagram M under a given interpretation is a scalar that is defined by the usual "summation over internal indices" formula. For example, the denotation of the above diagram N is:

$$[\![N]\!] = \sum_{a_1,b_2,b_3,a_4,a_5} [\![g]\!]^{a_1,b_2}_{b_3,a_1} \cdot [\![f]\!]^{b_3}_{a_5,a_4} \cdot [\![f^\dagger]\!]^{a_5,a_4}_{b_2}. \tag{4.1}$$

Here a_1, a_4, a_5 range over some orthonormal basis of $[\![A]\!]$, b_2, b_3 range over some orthonormal basis of $[\![B]\!]$, and $[\![f]\!]^{b_3}_{a_5,a_4}$ stands for the matrix entry $\langle a_5 \otimes a_4 \mid [\![f]\!](b_3) \rangle$. As is well-known, this denotation is independent of the choice of orthonormal bases.

4.2 Proof sketch

By the reductions in Section 3, Theorem 2.2 is a consequence of the following lemma:

Lemma 4.1 (Relative completeness) *Let M be a (closed simple dagger traced monoidal) diagram. Then there exists an interpretation $[\![-]\!]_M$ in finite dimensional Hilbert spaces, depending only on M, such that for all N, $[\![N]\!]_M = [\![M]\!]_M$ holds if and only if N and M are isomorphic diagrams.*

Clearly, the right-to-left implication is trivial, for if N and M are isomorphic diagrams, then $[\![N]\!] = [\![M]\!]$ holds under every interpretation; their corresponding summation formulas differ at most by a reordering of summands and factors. It is therefore the left-to-right implication that must be proved.

The general proof of this lemma requires quite a bit of notation, as well as more careful definitions than we have given above. A full proof will appear elsewhere. Here, we illustrate the proof technique by means of an example.

Take the same signature as above, and suppose M is the following diagram:
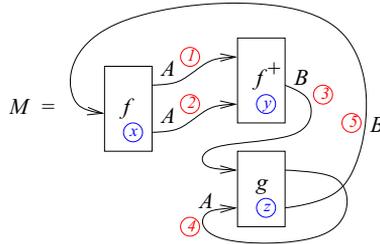
Again, we have numbered the wires from 1 to 5, and this time, we have also numbered the boxes x, y, and z.

We must now construct the interpretation required by the Lemma. It is given as follows. Define $[\![A]\!]_M$ to be a 3-dimensional Hilbert space with orthonormal basis $\{A_1, A_2, A_4\}$. Define $[\![B]\!]_M$ to be a 2-dimensional Hilbert space with orthonormal basis $\{B_3, B_5\}$. Note that the names of the basis vectors have been chosen to suggest a correspondence between basis vectors of $[\![A]\!]_M$ and wires labeled A in the diagram M, and similarly for $[\![B]\!]_M$.

Let x, y, and z be three algebraically independent transcendental complex numbers. This means that x, y, z do not satisfy any polynomial equation $p(x, y, z, \bar{x}, \bar{y}, \bar{z}) = 0$ with rational coefficients, unless $p \equiv 0$.

Define three linear maps $F_x : [\![B]\!]_M \to [\![A]\!]_M \otimes [\![A]\!]_M$, $F_y : [\![A]\!]_M \otimes [\![A]\!]_M \to [\![B]\!]_M$, and $F_z : [\![A]\!]_M \otimes [\![B]\!]_M \to [\![B]\!]_M \otimes [\![A]\!]_M$ as follows. We give each map by its matrix representation in the chosen basis.

$$(F_x)^i_{jk} = \begin{cases} x & \text{if } i = B_5, \ j = A_2, \text{ and } k = A_1, \\ 0 & \text{else,} \end{cases}$$

$$(F_y)^{ij}_k = \begin{cases} y & \text{if } i = A_2, \ j = A_1, \text{ and } k = B_3, \\ 0 & \text{else,} \end{cases}$$

$$(F_z)^{ij}_{kl} = \begin{cases} z & \text{if } i = A_4, \ j = B_3, \ k = B_5, \text{ and } l = A_4, \\ 0 & \text{else.} \end{cases}$$

It is hopefully obvious how each of these linear functions is derived from the diagram M: each matrix contains precisely one non-zero entry, whose position is determined by the numbering of the input and output wires of the corresponding box in M.

The interpretations of f and g are then defined as follows:

$$[\![f]\!]_M = F_x + F_y^\dagger, \qquad [\![g]\!]_M = F_z.$$

Note that we have taken the adjoint of the matrix F_y, due to the fact that the corresponding box was labeled $f^\dagger$. This finishes the definition of the interpretation $[\![-]\!]_M$. It can be done analogously for any diagram M.

To prove the condition of the Lemma, we first observe that the interpretation $[\![N]\!]_M$ of any diagram N is given by a summation formula analogous to (4.1). Moreover, from the definition of the interpretation $[\![-]\!]_M$, it immediately follows that the scalar $[\![N]\!]_M$ can be (uniquely) expressed as a polynomial $p(x, y, z, \bar{x}, \bar{y}, \bar{z})$ with integer coefficients in the variables x, y, z and their complex conjugates. We also note that this polynomial is homogeneous, and its degree is equal to the number of boxes in N.

We claim that the coefficient of p at xyz is non-zero if and only if N is isomorphic to M. The proof is a direct calculation, using (4.1) and the definition of $[\![-]\!]_M$. Essentially, any non-zero contribution to xyz in the summation formula must come from a choice of a basis vector $A_{\psi(w)}$ of $[\![A]\!]_M$ for each wire w labeled A in N, and a choice of a basis vector $B_{\psi(w)}$ of $[\![B]\!]_M$ for each wire w labeled B in N, together with a bijection ϕ between the boxes of N and the set $\{x, y, z\}$; moreover, the contribution can only be non-zero if the choice of basis vectors is "compatible" with the bijection ϕ. Compatibility amounts precisely to the requirement that the maps ψ and ϕ determine a graph isomorphism from N to M. For example, in the calculation of $[\![N]\!]_M$ according to equation (4.1), the only non-zero contribution to xyz in p comes from the assignment $a_1 \mapsto A_4$, $b_2 \mapsto B_3$, $b_3 \mapsto B_5$, $a_4 \mapsto A_1$, and $a_5 \mapsto A_2$, which corresponds exactly to the (in this case unique) isomorphism from N to M.

In fact, we get a stronger result: the integer coefficient of p at xyz is equal to the *number* of different isomorphisms between N and M (usually 0 or 1, but it could be higher if M has non-trivial automorphisms).

5   Generalizations

Other fields

The result of this paper (Theorem 2.2) can be adapted to other fields besides the complex numbers. It is true for any field k of characteristic 0 with a non-trivial involutive automorphism $x \mapsto \bar{x}$. (Non-trivial means that for some x, $\bar{x} \neq x$).

The only special property of $\mathbb{C}$ that was used in the proof, and which may not hold in a general field k, was the existence of transcendentals. This problem is easily solved by first considering the field of fractions $k(x_1, \ldots, x_n)$, where the required transcendentals have been added freely. The proof of Lemma 4.1 then proceeds without change. Finally, once an interpretation over $k(x_1, \ldots, x_n)$ has been found such that $[\![M]\!] \neq [\![N]\!]$, we use the fact that in a field of characteristic 0, any non-zero polynomial has a non-root. Thus we can instantiate $x_1, \ldots, x_n$ to specific elements of k while preserving the inequality $[\![M]\!] \neq [\![N]\!]$. Note that therefore, Theorem 2.2 holds for k; however, Lemma 4.1 only holds for $k(x_1, \ldots, x_n)$.

Bounded dimension

The interpretation $[\![-]\!]_M$ from Section 4.2 uses Hilbert spaces of unbounded dimension. One may ask whether Theorem 2.2 remains true if the dimension of the Hilbert spaces is fixed to some n. This is known to be false when $n = 2$. Here is

a counterexample due to Bob Paré: the equation $\text{tr}(AABBAB) = \text{tr}(AABABB)$ holds for all 2×2-matrices, but does not hold in the graphical language. Indeed, by the Cayley-Hamilton theorem, $A^2 = \mu A + \nu I$ for some scalars μ, ν. Therefore

$$\text{tr}(AABBAB) = \mu \, \text{tr}(ABBAB) + \nu \, \text{tr}(BBAB),$$

$$\text{tr}(AABABB) = \mu \, \text{tr}(ABABB) + \nu \, \text{tr}(BABB),$$

and the right-hand-sides are equal by cyclicity of trace. It is not currently known to the author whether Theorem 2.2 is true when restricted to spaces of dimension 3.

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