Research Article

Some Characteristic Properties of Parallel \(z\)-Equidistant Ruled Surfaces

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1. Introduction

The basis notions about ruled surfaces in \(E^3\) are given in [1]. Parallel \(p\)-equidistant ruled surfaces are described, and some of their characteristic properties are given in Valeontis’s article entitled “Parallel \(p\)-Aquidistante Regelflächen” [2]. Integral invariants, shape operators and spherical indicators of Parallel \(p\)-equidistant ruled surfaces were computed by Masal and Kuruoğlu in the articles [3–6]. Mannheim curves were described in Liu and Wang’s article entitled “Mannheim partner curve in 3-Space” [7]. Some characteristic properties of Mannheim curves and Mannheim offsets of ruled surfaces were studied in [8, 9].

2. Preliminaries

Let \(\alpha : I \rightarrow E^3\) be a differentiable curve with arc-length parameter \(s\), and \(\{u_1, u_2, u_3\}\) be the Frenet frame of \(\alpha\) at the point \(\alpha(s)\), where

\[
\begin{align*}
    u_1(s) &= \alpha'(s), \\
    u_2(s) &= \frac{\alpha''(s)}{\|\alpha''(s)\|}, \\
    u_3(s) &= u_1(s) \wedge u_2(s).
\end{align*}
\]

(1)

The Frenet formulas of \(\alpha\) are

\[
\begin{bmatrix}
    u_1' \\
    u_2' \\
    u_3'
\end{bmatrix} =
\begin{bmatrix}
    0 & k_1 & 0 \\
    -k_1 & 0 & k_2 \\
    0 & -k_2 & 0
\end{bmatrix}
\begin{bmatrix}
    u_1 \\
    u_2 \\
    u_3
\end{bmatrix}.
\]

(2)

If \(\alpha\) is a curve and \(x\) is a generator vector, then the ruled surface \(X(s, v)\) has the following parameter representation:

\[
X(s, v) = \alpha(s) + vx(s).
\]

(3)

Namely, a ruled surface is a surface generated by the motion of a straight line \(x\) along \(\alpha\). Furthermore, if \(\alpha\) is a closed curve, then this surfaces is called closed ruled surface. Moreover, the drall \(P_x\), the striction \(\gamma\), the apex angle \(\lambda_x\), and the pitch \(l_x\) of the closed ruled surface are defined by

\[
P_x = \text{det} \left( \alpha', x', x'' \right) / \|x'\|^2,
\]

\[
\gamma(s) = \alpha(s) - \frac{\left( x'(s), \alpha'(s) \right)}{\|x'(s)\|^2} x(s),
\]

\[
\lambda_x = \langle d, x \rangle, \quad l_x = \langle V, x \rangle.
\]

(4)
The angle of the pitches, pitches, and dralls of the closed ruled surface generated by the Frenet vectors \( u_1, u_2, \) and \( u_3 \) are
\[
\begin{align*}
\lambda_{u_1} &= \oint (u_1, k_2) ds, \\
\lambda_{u_2} &= \int k_2 ds, \\
\lambda_{u_3} &= \int k_1 ds,
\end{align*}
\]
so that
\[
\begin{align*}
l_{u_1} &= \frac{k_1}{k_2} (l_{u_1} + a_1), \\
l_{u_2} &= l_{u_2} = 0, \\
l_{u_3} &= l_{u_3} = 0,
\end{align*}
\]
see [3].

Definition 4. Consider two space curves \( \alpha : I \rightarrow E^3 \) and \( \beta : I \rightarrow E^3 \), where \( I \) is a real interval that has at least four continuous derivatives. If there exists a corresponding relationship between the space curves \( \alpha \) and \( \beta \) such that the principal normal lines of \( \alpha \) coincide with the binormal lines of \( \beta \) at the corresponding points of the curves, then \( \alpha \) is called as a Mannheim curve, and \( \beta \) is called as a Mannheim partner curve of \( \alpha \). The pair of \( \{\alpha, \beta\} \) is said to be a Mannheim pair [7].

Definition 5. Let \( T \) be the unit tangent vector of the curve, \( \alpha : I \rightarrow E^3 \). If \( T \) makes a constant angle with a fixed line, then the curve \( \alpha \) is called the helix curve [1].

Theorem 6. The distance between corresponding points of the Mannheim partner curves in \( E^3 \) is constant [8].

Theorem 7. For a curve \( \alpha \) in \( E^3 \), there is a curve \( \beta \), so that \( \{\alpha, \beta\} \) is a Mannheim pair [8].

Theorem 8. Let curvature and torsion of \( \alpha \) be \( k_1 \) and \( k_2 \), respectively, and \( \alpha \) is Mannheim curve \( \iff \lambda = k_1/(k_1^2 + k_2^2), \) (\( \lambda + 0 = \text{const} \)). See [7].

### 3. Parallel \( z \)-Equidistant Ruled Surfaces

Definition 9. Let \( \alpha \) and \( \overline{\alpha} \) be two curves, and let \( \{u_1, u_2, u_3\} \) and \( \{v_1, v_2, v_3\} \) be the Frenet frames of \( \alpha \) and \( \overline{\alpha} \) at the points \( \alpha(s) \) and \( \overline{\alpha}(s) \), respectively, in \( E^3 \). If the unit principal normal vectors, \( u_3(s) \) and \( v_3(s) \), are generator vectors and \( \alpha \) and \( \overline{\alpha} \) are anchor curve, then parametric equations of the two ruled surfaces are \( S(s, v) = \alpha(s) + v u_3(s) \) and \( \overline{S}(s, v) = \overline{\alpha}(s) + v v_3(s) \) in \( E^3 \). For this surfaces, if principal normal vectors \( u_2 \) and \( v_2 \) are parallel, and the distance \( z \) between central planes in suitable points are constant, then this couple ruled surface are called parallel \( z \)-equidistant ruled surfaces [9].

Let striction curve and curvatures of \( S \) ruled surface be \( \gamma(s), k_1(s) \), and \( k_2(s) \), respectively. Let striction curve and curvatures of \( \overline{S} \) ruled surface be \( \overline{\gamma}(s), k_1(s) \), and \( k_2(s) \), respectively. In this situation, parametric equations of striction curves are
\[
\begin{align*}
\gamma(s) &= \alpha(s) - \frac{\langle u'_1(s), \alpha'(s) \rangle}{\| u'_1(s) \|^2} u_2(s), \quad u'_2(s) \neq 0,
\overline{\gamma}(s) &= \overline{\alpha}(s) - \frac{\langle v'_1(s), \overline{\alpha}'(s) \rangle}{\| v'_1(s) \|^2} v_2(s), \quad v'_2(s) \neq 0.
\end{align*}
\]

If the Frenet formulas are written in the last equation, then we have
\[
\begin{align*}
\gamma(s) &= \alpha(s) - \frac{k_1(s)}{k_1(s) + k_2(s)} u_2(s), \\
\overline{\gamma}(s) &= \overline{\alpha}(s) - \frac{k_1(s)}{k_1(s) + k_2(s)} v_2(s).
\end{align*}
\]

If vector \( \gamma \) is written related to frame \( \{u_1, u_2, u_3\} \), we get
\[
\gamma = pu_1 + z u_2 + qu_3,
\]
where \( p = \langle \gamma, u_1 \rangle, z = \langle \gamma, u_2 \rangle, \) and \( q = \langle \gamma, u_3 \rangle \). Here, \( |p|, |z|, \) and \( |q| \) are distance between polar, central, and asymptotic planes, respectively.
**Theorem 10.** Let striction curves of $S$ and $\overline{S}$ parallel $z$-equidistant ruled surfaces be $\gamma$ and $\overline{\gamma}$. Then, the relation between striction curves are given as follows:

$$\overline{\gamma} = \gamma + pu_1 + q u_3 + \left( k_1 \left( 1 + \frac{k_1^2}{k_1^2 + k_2^2} \right) + p' \right) \frac{\vec{k}_1}{\vec{k}_1 + k_2^2} u_1$$

$$- k_2 \left( q' - \left( \frac{k_1 k_2}{k_1^2 + k_2^2} \right) \right) \times \left( k_1^2 + k_2^2 \right)^{-1} u_2. \quad (12)$$

**Proof.** By substituting (10) into (11), we get

$$\overline{\alpha} = \frac{\vec{k}_1}{k_1 + k_2^2} v_2 = \gamma + pu_1 + z u_2 + q u_3. \quad (13)$$

Since vectors $v_2$ and $u_2$ are parallel vectors, then we can write

$$\overline{\alpha} = \gamma + pu_1 + \left( z + \frac{\vec{k}_1}{k_1 + k_2^2} \right) u_2 + qu_3. \quad (14)$$

By differentiating the last equation, we have

$$\overline{\alpha}' = \left( 1 + \frac{k_1^2}{k_1^2 + k_2^2} \right) + p' - \left( z + \frac{\vec{k}_1}{k_1 + k_2^2} \right) k_1 u_1$$

$$+ k_2 \left( k_1 k_2 \right) u_2 + \left( z + \frac{\vec{k}_1}{k_1 + k_2^2} \right) \left( k_1^2 + k_2^2 \right)^{-1} u_3. \quad (15)$$

From the last equation, we have

$$\left\langle \overline{\alpha}', u_2' \right\rangle = -k_1 \left[ \left( 1 + \frac{k_1^2}{k_1^2 + k_2^2} \right) + p' \right]$$

$$+ k_2 \left[ -k_2 \left( k_1 k_2 \right) \right] + q'$$

$$+ \left( z + \frac{\vec{k}_1}{k_1 + k_2^2} \right) \left( k_1^2 + k_2^2 \right). \quad (16)$$

Substituting (14) and (16) into the last equation, then we obtain

$$\overline{\gamma} = \gamma + pu_1 + q u_3 + \left( k_1 \left( 1 + \frac{k_1^2}{k_1^2 + k_2^2} \right) + p' \right) \frac{\vec{k}_1}{\vec{k}_1 + k_2^2} u_1$$

$$- k_2 \left( q' - \left( \frac{k_1 k_2}{k_1^2 + k_2^2} \right) \right) \times \left( k_1^2 + k_2^2 \right)^{-1} u_2. \quad (18)$$

By (11), we have the following results.

**Corollary 11.** The distance between central planes of $S$ and $\overline{S}$ parallel $z$-equidistant ruled surfaces is

$$z = \lambda \left( 2 + p' - q k_2 \right). \quad (19)$$

**Corollary 12.** If $[\alpha, \gamma]$ and $[\overline{\alpha}, \overline{\gamma}]$ pairs are Mannheim pairs, the distance between central planes of $S$ and $\overline{S}$ parallel $z$-equidistant ruled surfaces is

$$z = \left( 2 + p' - q k_2 \right). \quad (20)$$

**Corollary 13.** Let the unit tangent vectors of the striction curves of surfaces $S$ and $\overline{S}$ be $T$ and $\overline{T}$, respectively. Let $\delta$ and $\overline{\delta}$ be the angles between vectors $T$, $\overline{T}$, and their projection vectors on central plane, respectively. In this case, the distance between central planes of parallel $z$-equidistant ruled surfaces $S$ and $\overline{S}$ can be obtained by the following equation (Figure 1):

$$z = k_1 \left( \cos \delta \sin \beta \frac{ds_y}{ds} + p' \right) - k_2 \left( \cos \delta \cos \beta \frac{ds_y}{ds} + q' \right)$$

$$\times \left( k_1^2 + k_2^2 \right)^{-1}. \quad (21)$$

**Proof.** By some algebraic manipulations, $T$ and $\overline{T}$ can be calculated as follows:

$$T = \left( \cos \delta \sin \beta u_1 + \sin \delta u_2 + \cos \delta \cos \beta u_3 \right) \frac{ds}{ds_y}, \quad (22)$$

$$\overline{T} = \left( \cos \overline{\delta} \sin \overline{\beta} v_1 + \sin \overline{\delta} v_2 + \cos \overline{\delta} \cos \overline{\beta} v_3 \right) \frac{ds}{ds_y}. \quad (23)$$
By differentiating (14), we have

\[
\alpha' = \left[ \cos \delta \sin \beta \frac{ds_y}{ds} + p' - \left( z + \left( \frac{k_1}{k_1 + k_2} \right) k_1 \right) \right] u_1 \\
+ \left[ \sin \gamma \frac{ds_x}{ds} + pk_1 - qk_2 + \left( z + \frac{k_1}{k_1 + k_2} \right) \right] u_2 \\
\times u_3 + \left[ \cos \delta \cos \beta \frac{ds_y}{ds} + q' \right] u_4 \\
+ \left( z + \frac{k_1}{k_1 + k_2} \right) k_2 \right] u_3,
\]  
(24)

From the last equation, we can write

\[
\langle \alpha', u_2' \rangle = \left( -k_1 \cos \delta \sin \beta - k_1 p' + k_2 \cos \delta \cos \beta \right) \\
\times \frac{ds_x}{ds} + k_2 q' + \left( z + \frac{k_1}{k_1 + k_2} \right) \\
\times \left( k_1^2 + k_2^2 \right) 
\]  
(25)

Substituting (14) and (23) into equation \( \bar{y} = \bar{\alpha} - (\langle u_2', \bar{\alpha}' \rangle/\|u_2'\|^2)u_2 \), we get

\[
\bar{y} = \gamma + pu_1 + qu_3 \\
+ \left[ \left( k_1 \left( \cos \delta \sin \beta \frac{ds_y}{ds} + p' \right) - k_2 \left( \cos \delta \cos \beta \frac{ds_y}{ds} + q' \right) \right) \right] u_2 \\
\times \left( k_1^2 + k_2^2 \right)^{-1} u_2, \]  
(26)

From (11), we have

\[
z = \left( k_1 \left( \cos \delta \sin \beta \frac{ds_y}{ds} + p' \right) - k_2 \left( \cos \delta \cos \beta \frac{ds_y}{ds} + q' \right) \right) \times \left( k_1^2 + k_2^2 \right)^{-1}. \]
(27)

**Corollary 14.** If \( \{\alpha, \gamma \} \) and \( \{\bar{\alpha}, \bar{\gamma} \} \) pairs are Mannheim pairs, then the distance between central planes of surfaces \( S \) and \( \bar{S} \) is

\[
z = \lambda \left[ \left( \cos \delta \sin \beta - \frac{k_2}{k_1} \cos \delta \cos \beta \right) \frac{ds_y}{ds} + p' - \frac{k_2}{k_1} q' \right]. \]
(28)

**Theorem 15.** Let \( S \) and \( \bar{S} \) be parallel \( z \)-equidistant ruled surfaces. Then, the relation between Frenet frame of \( \alpha, \{u_1, u_2, u_3\} \) and of \( \bar{\alpha}, \{v_1, v_2, v_3\} \) is given as follows:

\[
v_1 = \cos \varphi u_1 - \sin \varphi u_3, \]
(29)

\[
v_2 = u_2, \]
(30)

\[
v_3 = \sin \varphi u_1 + \cos \varphi u_3, \]
(31)

where \( \varphi \) is the angle between the vector \( v_1 \) and the vector \( u_1 \).

**Proof.** Let \( \varphi \) be the angle between the vector \( v_1 \) and the vector \( u_1 \). In this case, we can write

\[
v_1 = \cos \varphi u_1 + \sin \varphi u_3. \]
(32)

Since the vector \( v_2 \) is parallel to the vector \( u_2 \), we have

\[
v_2 = u_2, \]
(33)

\[
v_3 = v_1 \wedge v_2, \]
(34)

\[
v_3 = -\sin \varphi u_1 + \cos \varphi u_3. \]
(35)

This completes the proof of the theorem. \( \square \)

**Theorem 16.** Let \( S \) and \( \bar{S} \) be parallel \( z \)-equidistant ruled surfaces. Let \( s \) and \( \bar{s} \) be arc parameters of anchor curves of \( S \)
and $\mathcal{S}$, respectively. If $k_1$, $k_2$ and $\hat{k}_1$, $\hat{k}_2$ are curvatures of anchor curves of $S$ and $\mathcal{S}$, respectively, there are following equations between these curvatures:

$$\hat{k}_1 = (\cos \varphi k_1 - \sin \varphi k_2) \frac{ds}{d\hat{s}}, \quad (32)$$

$$\hat{k}_2 = (\sin \varphi k_1 + \cos \varphi k_2) \frac{ds}{d\hat{s}}, \quad (33)$$

Proof. Since $S$ and $\mathcal{S}$ parallel $z$-equidistant ruled surfaces, $u_2(s) = v_2(\hat{s})$. Differentiating this equation related to $s$, we have

$$-k_1 u_1 + k_2 u_3 = (\hat{k}_1 v_1 + \hat{k}_2 v_3) \frac{d\hat{s}}{ds}. \quad (34)$$

Multiplying the last equation with $v_1$ and $v_3$, we have

$$\hat{k}_1 = (\cos \varphi k_1 - \sin \varphi k_2) \frac{ds}{d\hat{s}}, \quad (35)$$

$$\hat{k}_2 = (\sin \varphi k_1 + \cos \varphi k_2) \frac{ds}{d\hat{s}}. \quad (36)$$

**Theorem 17. The relations between apex angles of closed parallel $z$-equidistant ruled surfaces $S$ and $\mathcal{S}$**

(1) $\lambda_{v_1} = \cos \varphi \lambda_{u_1} + \sin \varphi \lambda_{u_3} + b_1$, $b_1 = \oint_{(p + (z + (\hat{k}_1 i + \hat{k}_2 j))/u_1 + u_3)} \hat{k}_2 d\hat{s} + \oint_{(-k_1 i + k_2 j)/u_1 + u_3}) \hat{k}_2 d\hat{s}, \quad (37)$

(2) $\lambda_{v_2} = \lambda_{u_2} = 0$.

(3) $\lambda_{v_3} = -\sin \varphi \lambda_{u_1} + \cos \varphi \lambda_{u_3} + b_2$, $b_2 = \oint_{(p + (z + (\hat{k}_1 i + \hat{k}_2 j))/u_1 + u_3)} \hat{k}_1 d\hat{s} + \oint_{((-k_1 i + k_2 j)/u_1 + u_3) \hat{k}_1 d\hat{s}}, \quad (38)$

Proof. The apex angle of closed ruled surface which is generated by the unit tangent vector $v_1$ is $\lambda_{v_1} = \oint_{\hat{S}} \hat{k}_2 d\hat{s}$. Substituting (14) into the last equation, we get

$$\lambda_{v_1} = \oint_{(p + (z + (\hat{k}_1 i + \hat{k}_2 j))/u_1 + u_3)} \hat{k}_2 d\hat{s}, \quad (39)$$

$$\lambda_{v_1} = \oint_{(\hat{k}_1 i + \hat{k}_2 j)/u_1 + u_3) \hat{k}_2 d\hat{s}. \quad (40)$$

Impending,

$$b_1 = \oint_{(p + (z + (\hat{k}_1 i + \hat{k}_2 j))/u_1 + u_3)} \hat{k}_2 d\hat{s}$$

$$+ \oint_{((-k_1 i + k_2 j)/u_1 + u_3) \hat{k}_2 d\hat{s}}, \quad (38)$$

Impending,

$$\lambda_{v_1} = \oint_{(\hat{k}_1 i + \hat{k}_2 j)/u_1 + u_3) \hat{k}_2 d\hat{s} + b_1. \quad (39)$$

From (5), we can write

$$\lambda_{v_1} = \cos \varphi \lambda_{u_1} + \sin \varphi \lambda_{u_3} + b_1. \quad (40)$$

From (5), the apex angles of closed ruled surfaces which are generated with vector $v_2$ and $u_2$ are

$$\lambda_{v_2} = \lambda_{u_2} = 0. \quad (41)$$

The apex angle of closed ruled surface which is generated with vector $v_3$ is $\lambda_{v_3} = \oint_{\mathcal{S}} \hat{k}_1 d\hat{s}$. Substituting (14) into the last equation, we get

$$\lambda_{v_2} = \oint_{(\hat{k}_1 i + \hat{k}_2 j)/u_1 + u_3) \hat{k}_1 d\hat{s}, \quad (39)$$

$$\lambda_{v_2} = \oint_{((k_1 i + k_2 j)/u_1 + u_3) \hat{k}_1 d\hat{s}. \quad (40)$$

Substituting (9) into the last equation, we have

$$\lambda_{v_3} = \oint_{(\hat{k}_1 i + \hat{k}_2 j)/u_1 + u_3) \hat{k}_1 d\hat{s}, \quad (39)$$

$$\lambda_{v_3} = \oint_{((-k_1 i + k_2 j)/u_1 + u_3) \hat{k}_1 d\hat{s}. \quad (40)$$

Impending,

$$b_2 = \oint_{(p + (z + (\hat{k}_1 i + \hat{k}_2 j))/u_1 + u_3)} \hat{k}_1 d\hat{s}$$

$$+ \oint_{((-k_1 i + k_2 j)/u_1 + u_3) \hat{k}_1 d\hat{s}}, \quad (38)$$

Impending,

$$\lambda_{v_3} = \oint_{(\hat{k}_1 i + \hat{k}_2 j)/u_1 + u_3) \hat{k}_1 d\hat{s} + b_2. \quad (39)$$
Substituting (32) into the last equation, we have
\[ \lambda v_3 = -\sin \varphi \lambda u_1 + \cos \varphi \lambda u_2 + b_2. \] (45)

**Corollary 18.** If \{\alpha, \gamma\} and \{\alpha, \nu, \eta\} pairs are Mannheim pairs, then the relations between apex angles of closed parallel \( z \)-equidistant ruled surfaces \( S \) and \( \overline{S} \) are given as follows:

1. \[ \lambda v_1 = \cos \varphi \lambda u_1 + \sin \varphi \lambda u_2 + b_1, \quad b_1 = \oint_{(m, z + (\nu_i, \nu_j))} k_2 d\overline{s} + \oint_{(-m, z + (\nu_i, \nu_j))} k_2 d\overline{s}, \]
2. \[ \lambda v_2 = \lambda u_2 = 0, \]
3. \[ \lambda v_3 = -\sin \varphi \lambda u_1 + \cos \varphi \lambda u_2 + b_2, \quad b_2 = \oint_{(m, z + (\nu_i, \nu_j))} k_1 d\overline{s} + \oint_{(-m, z + (\nu_i, \nu_j))} k_1 d\overline{s}. \]

**Theorem 19.** If we specially take helix instead of anchor curve of closed parallel \( z \)-equidistant ruled surface, there is relation between \( l_1 \) and \( l_2 \) as follows:

\[ l_1 = \left( \cos \varphi \frac{k_1}{k_1} - \sin \varphi \frac{k_2}{k_2} \right) l_1 + b_3, \]
\[ b_3 = \oint_{(m, z + (\nu_i, \nu_j))} \left( \sin \varphi \frac{k_1}{k_1} \right) k_2 d\overline{s}, \]
\[ l_1 = \oint_{(y)} \sin \varphi \frac{k_1}{k_1} d\overline{s} + \oint_{(m, z + (\nu_i, \nu_j))} \left( \sin \varphi \frac{k_1}{k_1} \right) k_2 d\overline{s}. \] (46)

**Proof.** From (3), the pitch of closed ruled surface which is generated vector \( v_1 \) is \( l_1 = \oint_{(y)} d\overline{s} \). Substituting (14) into the last equation, we get

\[ l_1 = \oint_{(y)} d\overline{s} + \oint_{(m, z + (\nu_i, \nu_j))} \left( \sin \varphi \frac{k_1}{k_1} \right) k_2 d\overline{s}. \] (47)

Substituting (9) into the last equation, we get

\[ l_1 = \oint_{(y)} d\overline{s} + \oint_{(m, z + (\nu_i, \nu_j))} \left( \sin \varphi \frac{k_1}{k_1} \right) k_2 d\overline{s}, \]
\[ l_1 = \oint_{(y)} d\overline{s} + \oint_{(m, z + (\nu_i, \nu_j))} \left( \sin \varphi \frac{k_1}{k_1} \right) k_2 d\overline{s}. \] (48)

Impending,

\[ b_3 = \oint_{(m, z + (\nu_i, \nu_j))} \left( \sin \varphi \frac{k_1}{k_1} \right) k_2 d\overline{s}, \]
\[ l_1 = \oint_{(y)} d\overline{s} + b_3. \] (49)

Substituting (32) into this last equation, we have

\[ l_1 = \oint_{(y)} \left( \cos \varphi \frac{k_1}{k_1} - \sin \varphi \frac{k_2}{k_2} \right) d\overline{s} + b_3. \] (50)

Since anchor curve is helix curve, then curvatures are constant. In this situation, we have

\[ l_1 = \left( \cos \varphi \frac{k_1}{k_1} - \sin \varphi \frac{k_2}{k_2} \right) \oint_{(y)} d\overline{s} + b_3. \] (51)

Substituting (5) into the last equation, we obtain

\[ l_1 = \left( \cos \varphi \frac{k_1}{k_1} - \sin \varphi \frac{k_2}{k_2} \right) l_1 + b_3. \] (52)

**Theorem 20.** Let \{\alpha, \nu, \eta\} and \{\nu_1, \nu_2, \nu_3\} be Frenet frame of anchor curves of \( S \) and \( \overline{S} \) closed parallel \( z \)-equidistant ruled surfaces. Then, the relations between dralls of ruled surfaces \( S \) and \( \overline{S} \) are given as follows:

\[ P_{v_1} = P_{\alpha u_1} = 0, \]
\[ P_{v_2} = P_{\nu u_2} \left( \cos \varphi \frac{k_1}{k_2} \right) \frac{d\overline{s}}{d\overline{s}}, \]
\[ P_{v_3} = P_{\nu u_3} \frac{k_2}{k_1 + k_2}. \] (53)

**Proof.** From (5), the drall of closed ruled surface which is generated vector \( v_1 \) is

\[ P_{v_1} = 0. \] (54)

From (5), the drall of closed ruled surface which is generated vector \( v_2 \) is

\[ P_{v_2} = \frac{\overline{k}_2}{k_1 + k_2}. \] (55)

Substituting (32) and (33) into the last equation, we have

\[ P_{v_2} = \left( \sin \varphi k_1 + \cos \varphi k_2 \right) \frac{d\overline{s}}{d\overline{s}} \times \left( \left( \cos \varphi k_1 - \sin \varphi k_2 \right) \frac{d\overline{s}}{d\overline{s}} \right) \left( \frac{d\overline{s}}{d\overline{s}} \right)^{-1}, \]
\[ P_{v_2} = \left( \sin \varphi k_1 + \cos \varphi k_2 \right) \frac{d\overline{s}}{d\overline{s}} + \sin \varphi \frac{k_1}{k_2} \frac{d\overline{s}}{d\overline{s}}. \] (56)

From (5),

\[ P_{v_2} = P_{\nu u_3} \left( \cos \varphi + \sin \varphi \frac{k_1}{k_2} \right) \frac{d\overline{s}}{d\overline{s}}. \] (57)
The goal of closed ruled surface which is generated vector $v_3$ is $P_{v_3} = 1/\overline{k_2}$,

$$P_{v_3} = \frac{1}{k_2^2}$$

(58)

Substituting (5) and (33) into this last equation, we have

$$P_{v_3} = P_{u_3} \frac{k_2}{\sin \psi k_1 + \cos \psi k_2} \frac{d\overline{s}}{ds}.$$  

(59)

References


