Global Existence and Blow-Up for the Euler-Bernoulli Plate Equation with Variable Coefficients

1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with smooth boundary and $\Gamma = \partial \Omega$. We will consider the following initial-boundary value problem:

\[
\begin{aligned}
&u_{tt} + \mathcal{A}^2 u - \gamma \mathcal{A} u_t + a |u_t|^{m-2} u_t = b |u|^{p-2} u, \\
&\quad x \in \Omega, \quad t \in (0, T), \\
&\quad u = \frac{\partial u}{\partial n} = 0, \quad x \in \Gamma, \quad t \in (0, T), \\
&\quad u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega,
\end{aligned}
\]

(1)

where $\mathcal{A} = \frac{\partial}{\partial n}$ is the so-called conormal derivative and $n = (\nu_1, \nu_2, \ldots, \nu_n)$ is the unit normal of $\Gamma$ pointing towards the exterior of $\Omega$ and $\nu_d = \mathcal{A} \nu$.

In physical terms the entries $a_{ij}(x)$ are related to coefficients of elasticity. Let $A(x) = (a_{ij}(x))$ be an $n \times n$ matrix for $x \in \mathbb{R}^n$, and let $x = (x_1, x_2, \ldots, x_n)$ be the natural coordinate system. For each $x \in \mathbb{R}^n$, we define the inverse matrix of $A(x)$ by $(g_{ij}(x)) = (a_{ij}(x))^{-1}$ and the inner product and norm over the tangent space $R^n_\nu = R^n$ by

\[
\begin{aligned}
&g(X, Y) = \langle X, Y \rangle_g = \sum_{i,j=1}^n g_{ij} \alpha_i \beta_j, \\
&\quad |X|_g = \langle X, X \rangle_g^{1/2}, \quad X = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}, \quad Y = \sum_{i=1}^n \beta_i \frac{\partial}{\partial x_i}.
\end{aligned}
\]

(4)

Then $(\mathbb{R}^n, g)$ is a Riemannian manifold with metric $g$. Denote the gradient operator in metric $g$ by $\nabla_g$. Then we have

\[
\nabla_g u = \sum_{i,j=1}^n \left( a_{ij}(x) \frac{\partial u}{\partial x_i} \right) \frac{\partial}{\partial x_j},
\]

(5)

\[
|\nabla_g u|^2_g = \langle \nabla_g u, \nabla_g u \rangle_g = \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}.
\]

(6)

It is well known that the equation

\[
\begin{aligned}
&u_{tt} + b |u_t|^{m-2} u_t = \Delta u + a |u|^{p-2} u
\end{aligned}
\]


\[
\begin{aligned}
&u_{tt} + \mathcal{A}^2 u - \gamma \mathcal{A} u_t + a |u_t|^{m-2} u_t = b |u|^{p-2} u, \\
&\quad x \in \Omega, \quad t \in (0, T), \\
&\quad u = \frac{\partial u}{\partial n} = 0, \quad x \in \Gamma, \quad t \in (0, T), \\
&\quad u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega,
\end{aligned}
\]

(1)
solution with appropriate boundary and initial conditions has a limit as $t$ goes to infinity. Avalos et al. [20] are interested in the case of thermoelastic plate and they established stability of the rest state. Eden and Milani [21] discussed the exponential attractors for extensible beam equation. Note that [24] differs from [20, 21] because [24] included a viscous term $\Delta u_t$ which played an important role in their consideration. Guzmá [22] considered the system of plate equation with damping and proved that the solution decays exponentially if the damping term behaves like a linear function, whereas the decay is of a polynomial order otherwise. Li and Wu [25] discussed the plate stabilization problem with infinite damping and showed that the energy of the problem decays exponentially provided that the negative damping is sufficiently small. For the Cauchy problem of multidimensional generalized double dispersion equation, Xu and Liu [27] proved the existence and nonexistence of global weak solution by potential well method.

Our purpose in this paper is to give the local existence, blow-up, global existence, and stability of the solution to the initial-boundary value problem (1).

We will write $\|\cdot\|$ denoting the usual $L^2(\Omega)$ norm $\|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_p$ denoting the usual $L^p(\Omega)$ norm $\|\cdot\|_{L^p(\Omega)}$ and denoting

$$
\int_\Omega \langle \nabla y, \nabla g \rangle \, dx = \| \nabla g \|_p^2.
$$

Let

$$
W = \left\{ u \in H^2(\Omega) \mid u = \frac{\partial u}{\partial y} = 0 \text{ on } \Gamma \right\},
$$

$$
\mathcal{H} = \left\{ u \in H^4(\Omega) \cap W \mid \mathcal{A} u = \frac{\partial \mathcal{A} u}{\partial y} = 0 \text{ on } \Gamma \right\}.
$$

By a weak solution of system (1) we mean a function $u : [0, T) \to W$ satisfying

$$
\frac{d}{dt} \int_\Omega u_t w \, dx + \int_\Omega \mathcal{A} u_t \mathcal{A} w \, dx + \gamma \int_\Omega \langle \nabla u_t, \nabla g \rangle \, dx
$$

$$
+ a \int_\Omega |u_t|^{m-2} u_t w \, dx = b \int_\Omega |u|^{p-2} u w \, dx,
$$

$$
u(0) = u_0, \quad u_t(0) = u_1,
$$

for any $w \in W$.

Our paper is organized as follows. In Section 2, we prove the local existence of the solution to the initial-boundary value problem (1). Section 3 contains the statements and the proof of the blow-up of the solution to problem (1) with $\gamma = 0$. Section 4 is devoted to the blow-up result for problem (1) with $m = 2$. In Section 5, we prove the global existence of the solution for problem (1). The last section is devoted to the asymptotic stability of the solution for problem (1).

2. Local Existence Result

In this section we establish a local existence result for the solution to problem (1) under suitable conditions on $m$ and $p$.

First, we give the following local existence result.
Theorem 1 (local existence). Suppose that
\[ p \geq 2, \quad \text{for } n \leq 4, \]
\[ 2 \leq p \leq \frac{2n - 2}{n - 4}, \quad \text{for } n > 4, \]
\[ m \geq 2, \quad \text{for } n \leq 4, \]
\[ 2 \leq m \leq \frac{2n - 2}{n - 4}, \quad \text{for } n > 4. \]
\[ (10) \]
\[
\begin{align*}
\left\| \frac{\partial u}{\partial y} \right\|_{L^p(\Omega)}^p + \left\| \frac{\partial u}{\partial y} \right\|_{L^m(\Omega \times (0, T))}^m & \leq C^1 \left\| \frac{\partial u}{\partial y} \right\|_{L^2(\Omega \times (0, T))}^p \left\| \frac{\partial u}{\partial y} \right\|_{L^m(\Omega \times (0, T))}^m \left\| \frac{\partial u}{\partial y} \right\|_{L^2(\Omega \times (0, T))}^m \leq R^2, \quad (21) \end{align*}
\]

Then for initial data \((u_0, u_1) \in W \times L^2(\Omega)\), there exists a unique weak solution of problem (I) satisfying
\[ u \in C([0, T], W), \]
\[ u_t \in C\left( [0, T], L^2(\Omega) \right) \cap L^m(\Omega \times (0, T)) \]
for \( T > 0 \) small enough.

Proof.

Step 1. For \( \nu(x, t) \) given, we consider the local existence of the problem
\[ u_{tt} + \partial^2 u - \gamma \partial_x u_t + a \left| u_t \right|^{m-2} u_t = b \left| \nu \right|^{p-2} \nu, \]
\[ x \in \Omega, \quad t \in (0, T), \]
\[ u = \frac{\partial u}{\partial y} = 0, \quad x \in \Gamma, \quad t \in (0, T), \]
\[ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega. \]
\[ (13) \]
We take sequences \( \{u_t^n\}, \{u^n\} \in C^{\infty}_0(\Omega) \) to approximate \( u_0 \) and \( u_1 \), respectively, and take a sequence \( \{\nu^n\} \in C([0, T], C^{\infty}_0(\Omega)) \) to approximate \( \nu \). Then we consider the problems
\[ u_{tt}^n + \partial^2 u^n - \gamma \partial_x u_t^n + a \left| u_t^n \right|^{m-2} u_t^n = b \left| \nu^n \right|^{p-2} \nu^n, \]
\[ x \in \Omega, \quad t \in (0, T), \]
\[ u_t^n = \frac{\partial u^n}{\partial y} = 0, \quad x \in \Gamma, \quad t \in (0, T), \]
\[ u^n(x, 0) = u_0^n, \quad u_t^n(x, 0) = u_1^n, \quad x \in \Omega. \]
\[ (14) \]
Using the same arguments as in [28], we get the existence of a sequence of unique solutions \( \{u^n, u_t^n\} \) of (14) satisfying
\[ u^n \in L^{\infty}((0, T), \mathcal{H}), \]
\[ u_t^n \in L^{\infty}((0, T), W) \cap L^m(\Omega \times (0, T)), \]
\[ u_{tt}^n \in L^{\infty}((0, T), L^2(\Omega)). \]
\[ (15) \]
For \( R > 0 \) large and \( T > 0 \), we let \( Z_{T,R} \) denote the set of all functions \( \nu \) which satisfy
\[ \nu \in C([0, T], W), \]
\[ u_t \in C\left( [0, T], L^2(\Omega) \right) \cap L^m(\Omega \times (0, T)), \]
\[ \nu(x, 0) = u_0(x), \quad \nu_t(x, 0) = u_1(x), \quad x \in \Omega, \]
\[ h(\nu(t)) := \left\| u_t \right\|_{L^2(\Omega \times (0, T))}^2 + \left\| \nu \right\|_{L^m(\Omega \times (0, T))}^2 \leq R^2, \quad (16) \]
It follows from the trace theorem that \( Z_{T,R} \) is nonempty if \( R \) is sufficiently large and \( T \) is small. For example, if we let
\[ \nu(x, t) = u_0(x) \cos t + u_1(x) \sin t, \quad (17) \]
then \( \nu \in Z_{T,R} \) for \( R \) suitable large and \( T \) small. In the present section, we always make this assumption.

Step 2. We proceed to show that the sequence \( \{u^n, u_t^n\} \) is Cauchy in \( Z_{T,R} \). For this aim, we set
\[ U := u^n - u, \quad V := \nu^n - \nu, \quad (18) \]
and then \( U \) satisfies
\[ U_{tt} + \partial^2 U - \gamma \partial_x U_t + a |u_t^n|^{m-2} u_t^n - a |u_t|^{m-2} u_t^n = b \left| \nu^n \right|^{p-2} \nu^n, \]
\[ x \in \Omega, \quad t \in (0, T), \]
\[ U = \frac{\partial U}{\partial y} = 0, \quad x \in \Gamma, \quad t \in (0, T), \]
\[ U(x, 0) = U_0(x) = u_0^n - u_0, \quad U_t(x, 0) = U_1(x) = u_t^n - u_t, \quad x \in \Omega. \]
\[ (19) \]
We multiply the equation of (19) by \( U_t \) and integrate over \( \Omega \times (0, t) \); then it follows that
\[ \int_0^t \int_\Omega \left( U_{tt}^2 + (\partial^2 U)^2 \right) dx + 2y \int_0^t \int_\Omega |\nabla U|^2 dx ds \]
\[ + 2a \int_0^t \int_\Omega \left( |u_t^n|^m - |u_t|^m \right) \left( u_t^n - u_t \right) dx ds = \int_0^t \int_\Omega \left( U_{tt}^2 + (\partial U)^2 \right) dx \]
\[ + 2b \int_0^t \int_\Omega \left( |\nu^n|^p - |\nu|^p \right) \left( \nu^n - \nu \right) U_t dx ds. \]
\[ (20) \]
To estimate from the above second term on the right side of (20), we use the inequality
\[ \left\| \nu^n \right\|^p \left| \nu^n - \nu \right| \left\| \nu \right\|^p \left| \nu^n \right|^p \leq C \left| \nu^n - \nu \right| \left( \left\| \nu^n \right\|^p - \left\| \nu \right\|^p \right) \]
\[ (21) \]
for \(v^\prime, v^\prime \in R\) and \(p \geq 2\). Note that \(C_i (i \geq 1)\) denote positive constants depending only on \(p, m, \gamma, a, b, \) and \(\Omega\) rather than the initial data \((u_0, u_t)\) in this section. To estimate the second term on the left side of (20), we use the inequality
\[
(\lvert u \rvert^{m-2} - \lvert \bar{u} \rvert^{m-2} \bar{w}) (w - \bar{w}) \geq C_2 \lvert w - \bar{w} \rvert^m
\]
for \(w, \bar{w} \in R\) and \(m \geq 2\). Then (20)–(22) yield
\[
\int_{\Omega} \left( U_i^2 + (\mathcal{A} U)^2 \right) dx + 2\gamma \int_0^T \int_{\Omega} \left\lVert \nabla_u U_i \right\rVert^2_g dx ds
+ 2aC_2 \int_0^T \int_{\Omega} \left\lVert u_i' - u_i'' \right\rVert^m dx ds
\leq \int_{\Omega} \left( U_i^2 + (\mathcal{A} U_0)^2 \right) dx
+ 2bC_1 \int_0^T \int_{\Omega} \left( \left\lvert v_i' \right\rvert^{p-2} - \left\lvert v_i'' \right\rvert^{p-2} \right) \bar{w} dx ds.
\]
(23)

From Hölder’s inequality and the Sobolev embedding theorem the last term of (23) takes the form
\[
I := 2bC_1 \int_0^T \int_{\Omega} \left\lvert v_i' - v_i'' \right\rvert \left\lVert U_i \right\rVert \left( \left\lvert v_i' \right\rvert^{p-2} - \left\lvert v_i'' \right\rvert^{p-2} \right) dx ds
\leq 2bC_3 \int_0^T \left\lVert U_i \right\rVert \left( \left\lVert v_i' \right\rVert_{n(p-2)/2} + \left\lVert v_i'' \right\rVert_{n(p-2)/2} \right) dx ds.
\]
(24)

Using Young’s inequality, we have
\[
I \leq 2bC_4 R^{p-2} \int_0^T \left( \left\lVert U_i \right\rVert^2 + \left\lVert \mathcal{A} V \right\rVert^2 \right) ds
\leq 2bC_4 R^{p-2} \int_0^T \left\lVert U_i \right\rVert^2 ds + 2bC_4 R^{p-2} T \max_{t \in [0, T]} \left\lVert \mathcal{A} V \right\rVert^2.
\]
(25)

Combining (23) and (25), we obtain, for any \(t \in [0, T]\),
\[
\left\lVert U_i \right\rVert^2 + \left\lVert \mathcal{A} U \right\rVert^2 + 2\gamma \left\lVert \nabla_g U_i \right\rVert_{g L^2((0, T) \times \Omega)}^2
+ 2aC_2 \left\lVert U_i \right\rVert^m_{L^m((0, T) \times \Omega)}
\leq \left\lVert U_i \right\rVert^2 + \left\lVert \mathcal{A} U_0 \right\rVert^2 + 2bC_4 R^{p-2} T \max_{t \in [0, T]} \left\lVert \mathcal{A} V \right\rVert^2
+ 2bC_4 R^{p-2} \int_0^T \left\lVert U_i \right\rVert^2 ds.
\]
(26)

It follows from (26) and Gronwall's inequality that, for any \(t \in [0, T]\),
\[
\left\lVert U_i \right\rVert^2 + \left\lVert \mathcal{A} U \right\rVert^2 + 2\gamma \left\lVert \nabla_g U_i \right\rVert_{g L^2((0, T) \times \Omega)}^2
+ 2aC_2 \left\lVert U_i \right\rVert^m_{L^m((0, T) \times \Omega)}
\leq \left[ \left\lVert U_i \right\rVert^2 + \left\lVert \mathcal{A} U_0 \right\rVert^2 + 2bC_4 R^{p-2} T \max_{t \in [0, T]} \left\lVert \mathcal{A} V \right\rVert^2 \right] e^{2bC_i R^{p-2} T}.
\]
(27)

Furthermore, we have
\[
\max_{t \in [0, T]} \left\lVert U_i \right\rVert^2 + \max_{t \in [0, T]} \left\lVert \mathcal{A} U \right\rVert^2 + 2\gamma \left\lVert \nabla_g U_i \right\rVert_{g L^2((0, T) \times \Omega)}^2
+ 2aC_2 \left\lVert U_i \right\rVert^m_{L^m((0, T) \times \Omega)}
\leq \left[ \left\lVert U_i \right\rVert^2 + \left\lVert \mathcal{A} U_0 \right\rVert^2 + 2bC_4 R^{p-2} T \max_{t \in [0, T]} \left\lVert \mathcal{A} V \right\rVert^2 \right] e^{2bC_i R^{p-2} T}.
\]
(28)

Since \(\{v^i\}, \{u_i^0\}\), and \(\{u_i^t\}\) are Cauchy in \(C([0, T]; W), W,\) and \(L^2(\Omega)\), respectively, we conclude that \(\{u^i\}, \{u_i^0\}, \{\nabla_g u_i^0\}, \) and \(\{u_i^t\}\) are Cauchy in \(C([0, T]; W), C([0, T]; L^2(\Omega)), L^2((0, T) \times \Omega), \) and \(L^m((0, T) \times \Omega)\), respectively.

**Step 3.** We now prove that the limit \((u(x, t), u_t(x, t))\) is a weak solution of (13).

To this end, we multiply equation of (14) by \(w \in W\) and integrate over \(\Omega\); then we obtain that
\[
\frac{d}{dt} \int_{\Omega} u_i^\prime w dx + \int_{\Omega} \mathcal{A} u_i^\prime w dx + \gamma \int_{\Omega} \langle \nabla_g u_i^\prime, \nabla_g w \rangle g dx
+ a \int_{\Omega} \left\lvert u_i^\prime \right\rvert^{m-2} u_i^\prime w dx = b \int_{\Omega} \left\lvert v_i^\prime \right\rvert^{p-2} v_i^\prime w dx.
\]
(29)

As \(\mu \rightarrow \infty\), the following hold:
\[
\int_{\Omega} \mathcal{A} u_i^\prime w dx \rightarrow \int_{\Omega} \mathcal{A} u w dx, \quad \text{in } C([0, T]),
\int_{\Omega} \left\lvert v_i^\prime \right\rvert^{p-2} v_i^\prime w dx \rightarrow \int_{\Omega} \left\lvert v \right\rvert^{p-2} v w dx, \quad \text{in } C([0, T]),
\int_{\Omega} \langle \nabla_g u_i^\prime, \nabla_g w \rangle g dx \rightarrow \int_{\Omega} \langle \nabla_g u, \nabla_g w \rangle g dx, \quad \text{in } C([0, T]),
\int_{\Omega} \left\lvert u_i^\prime \right\rvert^{m-2} u_i^\prime w dx \rightarrow \int_{\Omega} \left\lvert u \right\rvert^{m-2} u w dx, \quad \text{in } L^1((0, T)).
\]
(30)

Then it follows that
\[
\int_{\Omega} u_t w dx = \lim_{\mu \rightarrow \infty} \int_{\Omega} u_i^\prime w dx
\]
(31)
is an absolutely continuous function on \([0, T]\); thus for almost all \(t \in [0, T]\), \((u(x, t), u_t(x, t))\) is a weak solution of...
problem (13). To prove uniqueness, we denote that \( u^\mu, u^\nu \) are the corresponding solutions of problem (13) to \( v^\mu, v^\nu \), respectively. Then \( U = u^\mu - u^\nu \) satisfies

\[
\int_\Omega \left( U^2 + (\mathcal{A}U)^2 \right) dx + 2\gamma \int_0^t \int_\Omega |\nabla g U|_g^2 dx ds \\
+ 2aC_2 \int_0^t \int_\Omega |u^\mu - u^\nu|^m dx ds
\]

\[
\leq 2bC_1 \int_0^t \int_\Omega |v^\mu - v^\nu| |U|_g (|v^\mu|^{p-2} - |v^\nu|^{p-2}) dx ds.
\]  

(32)

This shows that \( U = 0 \) for \( v^\mu = v^\nu \). The uniqueness follows.

**Step 4.** We denote by \( \mathcal{K} \) the map which carries \( v \in Z_{T,R} \) into \( u \); that is,

\[
\mathcal{K} v = u,
\]  

(33)

where \((u, u_t)\) is the solution of problem (13). We establish an a priori estimate below to show that \( \mathcal{K} \) maps \( Z_{T,R} \) into itself if \( R \) is sufficiently large and \( T \) is sufficiently small relative to \( R \). We then equip \( Z_{T,R} \) with the complete metric \( \rho \) defined by

\[
\rho (w, \overline{w}) = \sup_{t \in [0,T]} h (w(t) - \overline{w}(t))
\]  

(34)

and show that \( \mathcal{K} \) is strict contraction if \( T \) is sufficiently small. The contraction mapping principle thus implies that \( \mathcal{K} \) has a unique fixed point which is obviously a solution to problem (1). For this purpose, we multiply the equation of (13) by \( 2u_t \) and integrate over \( \Omega \times (0, T) \) to get that

\[
\int_\Omega \left( U^2 + (\mathcal{A}U)^2 \right) dx + 2\gamma \int_0^t \int_\Omega |\nabla g U|_g^2 dx ds \\
+ 2a \int_0^t \int_\Omega |u|^m dx ds
\]

\[
= \int_\Omega \left( U^2 + (\mathcal{A}U)^2 \right) dx + 2b \int_0^t \int_\Omega |v|^{p-2} u_t dx ds.
\]  

(35)

From Hölder’s inequality, it follows that, for \( t \in [0,T] \),

\[
\int_\Omega \left( U^2 + (\mathcal{A}U)^2 \right) dx + 2\gamma \int_0^t \int_\Omega |\nabla g U|_g^2 dx ds \\
+ 2a \int_0^t \int_\Omega |u|^m dx ds
\]

\[
\leq \int_\Omega \left( U^2 + (\mathcal{A}U)^2 \right) dx + 2b \int_0^t \int_\Omega |u| |v|^{p-1} dx ds
\]

\[
\leq \int_\Omega \left( U^2 + (\mathcal{A}U)^2 \right) dx + 2bC_2 \int_0^t \int_\Omega |u| |v|^{p-1} dx ds
\]

\[
\leq \int_\Omega \left( U^2 + (\mathcal{A}U)^2 \right) dx + 2bC_2 \int_0^t \int_\Omega |u| |v|^{p-1} dx ds
\]

By choosing \( R \) large enough and then \( T \) sufficiently small, we obtain

\[
u \in Z_{T,R}.
\]  

(36)

This shows that \( \mathcal{K} \) maps \( Z_{T,R} \) into itself.

**Step 5.** We verify that \( \mathcal{K} \) is a contraction if \( T \) is sufficiently small. Let \( \nu, \overline{\nu} \in Z_{T,R} \), and set \( u = \mathcal{K} \nu \) and \( \overline{u} = \mathcal{K} \overline{\nu} \). Clearly, \( U := u - \overline{u} \) is the solution of the problem

\[
U + \mathcal{A}U + \gamma \mathcal{A}U_t + a |u|^m \gamma u - a \overline{|u|^m} \overline{\gamma} \overline{u}_t
\]

\[
= b |v|^{p-2} u - b |\overline{\nu}|^{p-2} \overline{\nu} \quad x \in \Omega, \ t \in (0, T),
\]  

(38)

\[
U = \frac{\partial U}{\partial \nu} = 0, \quad x \in \Gamma, \ t \in (0, T),
\]

\[
U(x, 0) = 0, \quad U_t(x, 0) = 0, \quad x \in \overline{\Omega}.
\]

Multiplying the equation of (38) by \( 2U_t \) and integrating it over \( \Omega \), we have

\[
\frac{d}{dt} \left[ \|U_t\|^2 + \|\mathcal{A}U_t\|^2 + 2\gamma \int_0^t \int_\Omega |\nabla g U|_g^2 ds \right] \\
+ 2a \int_0^t \int_\Omega \left( |u|^m - |\overline{u}|^m \right) (u_t - \overline{u}_t) dx ds
\]

\[
= 2b \int_\Omega U_t \left( |v|^{p-2} - |\overline{\nu}|^{p-2} \right) \overline{\nu} dx.
\]  

(39)

Noticing \( U(x, 0) = U_t(x, 0) = 0 \) and using the same arguments as (20)–(28), we get the estimate

\[
\rho (u, \overline{u}) \leq 2bC_6 R^p T e^{2bC_6 R^{p-1} T} \rho (v, \overline{\nu}),
\]  

(40)

where \( C_6 \) is a positive constant independent of \( R \) and \( T \). By choosing \( T \) so small that

\[
2bC_6 R^p T e^{2bC_6 R^{p-1} T} < 1,
\]  

(41)

then the map \( \mathcal{K} \) is a contraction. By the contraction mapping principle, the map \( \mathcal{K} \) has a unique fixed point which is obviously a solution \( u = \mathcal{K} \nu \in Z_{R,T} \). It is clear that \((u, u_t)\) is the desired solution of problem (1), and the proof of Theorem 1 is completed. \( \square \)
In the following we denote the maximal existence time of the solution by \( T_{\text{max}} = T \) and the energy of system (1) by
\[
E(t) = \frac{1}{2} \| u_t \|^2 + \frac{1}{2} \| \mathcal{A} u \|^2 - \frac{b}{p} \| u \|^p. \tag{42}
\]

Multiplying the equation of (1) by \( u_t \) and integrating over \( \Omega \), we obtain, for \( t \in [0, T) \),
\[
\frac{d}{dt} E(t) = -a \| u_t \|_m^m - \gamma \| \nabla_g u_t \|_g^2 \leq 0. \tag{43}
\]

Denote \( B > 0 \) is the best embedding constant such that
\[
\| w \|_p \leq B \| \Delta w \|, \quad \text{for } w \in W. \tag{44}
\]

Next we present two lemmas which will be used in the following sections.

**Lemma 2.** Suppose that \( p \) satisfies (10); then one has
\[
\| u \|_p^s \leq \left( \frac{B^2}{\lambda} + 1 \right) \left[ \| \mathcal{A} u \|^2 + \| u \|_p^2 \right] \tag{45}
\]
for \( s \in [2, p] \).

**Proof.** From the fact that \( \| u \|_p \leq B \| \Delta u \| \), we get
\[
\| u \|_p^s \leq \| u \|_p^2 \leq B^2 \| \Delta u \|^2 \leq \frac{B^2}{\lambda} \| \mathcal{A} u \|^2, \tag{46}
\]
if \( \| u \|_p \leq 1 \). Otherwise,
\[
\| u \|_p^s \leq \| u \|_p^2. \tag{47}
\]

Lemma 2 follows from (46) and (47).

**Lemma 3.** Suppose that the conditions of Theorem 1 hold, and let \( u(x, t) \) be the solution of problem (1) with the initial data satisfying \( E(0) < E_1 \) and \( \| \mathcal{A} u_0 \| > s_1 \), where
\[
E_1 := \frac{p-2}{2p} s_1^2, \quad s_1 := \left( \frac{\lambda p^2}{bB^p} \right)^{(p-2)/p}. \tag{48}
\]

Then there exists \( s_2 > s_1 \), such that
\[
\| \mathcal{A} u \| \geq s_2. \tag{49}
\]

**Proof.** By Sobolev-Poincaré inequality and the property of the operator \( \mathcal{A} \), we have
\[
\| u \|_p^p \leq B^p \| \Delta u \|^p \leq \frac{B^p}{\lambda p^2} \| \mathcal{A} u \|^p. \tag{50}
\]
Thus we get
\[
E_1 > E(0) \geq E(t) \geq \frac{1}{2} \| \mathcal{A} u \|^2 - \frac{b}{p} \| u \|^p \tag{51}
\]
\[
\geq \frac{1}{2} \| \mathcal{A} u \|^2 - \frac{bB^p}{\lambda p^2} \| \mathcal{A} u \|^p := F(\| \mathcal{A} u \|),
\]
where
\[
F(s) = \frac{1}{2} s^2 - \frac{bB^p}{\lambda p^2} s^p. \tag{52}
\]
It is easy to verify that the function \( F(s) \) has a maximum at \( s_1 = \left( \frac{\lambda p^2}{bB^p} \right)^{1/(p-2)} \) and the maximum value is \( E_1 = (p/2) s_1^2 \). From the definition of \( F(s) \), we see that \( F(s) \) is increasing in \((0, s_1)\) and decreasing in \((s_1, +\infty)\) and \( F(s) \to -\infty \) as \( s \to +\infty \). From the assumptions \( E(0) < E_1 \) and \( \| \mathcal{A} u_0 \| > s_1 \), we know that there exists \( s_2 > s_1 \), such that
\[
\| \mathcal{A} u \| \geq s_2, \tag{53}
\]
which completes the proof of Lemma 3.

3. Blow-Up Result for \( \gamma = 0 \)

In this section we establish the blow-up result for the following problem by the energy compensation method:
\[
\begin{align*}
& u_{tt} + \mathcal{A}^2 u + a |u_t|^{m-2} u_t = b |u|^{p-2} u, \\
& x \in \Omega, \quad t \in (0, T), \\
& u = \frac{\partial u}{\partial y_d} = 0, \quad x \in \Gamma, \quad t \in (0, T), \\
& u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega.
\end{align*} \tag{54}
\]

Our technique of proof follows closely the argument of [7] with the modifications needed for our problem. Denoting the energy of system (54) by
\[
E(t) = \frac{1}{2} \| u_t \|^2 + \frac{1}{2} \| \mathcal{A} u \|^2 - \frac{b}{p} \| u \|^p, \tag{55}
\]
multiplying the equation of (54) by \( u_t \), and integrating over \( \Omega \), we obtain, for \( t \in [0, T) \),
\[
\frac{d}{dt} E(t) = -a \| u_t \|_m^m. \tag{56}
\]

Next we give the blow-up result.

**Theorem 4.** Assume that the conditions of Theorem 1 hold, \( m < p \), and \( u(x, t) \) is the solution of (54) with the initial data
Let \((u_0, u_1) \in W \times L^2(\Omega)\) satisfying either one of the following conditions:

(i) \(E(0) < 0\);
(ii) \(0 \leq E(0) < E_1\) and \(\|Au_0\| > s_1\), in which

\[
E_1 := \frac{P - 2}{2p} s_1^2, \quad s_1 := \left(\frac{\lambda^{p/2}}{BB_0}\right)^{1/(p-2)}
\]

and \(B\) is the embedding constant satisfying

\[
\|u\|_p \leq B \|Au\|.
\]

Then the solution \(u(x, t)\) blows up in finite time.

**Proof.** In the following, we discuss two cases.

**Case 1** \((0 \leq E(0) < E_1)\). We set

\[
H(t) = M - E(t),
\]

where \(M > E(0)\) can be chosen later. Since

\[
H'(t) = a \|u_t\|_m^m \geq 0
\]

and \(H(0) = M - E(0) > 0\), hence

\[
0 < H(0) \leq H(t).
\]

Next we define

\[
G(t) = H_\alpha(t) + \epsilon \int_{\Omega} u_t \, dx,
\]

where \(\alpha, \epsilon\) are to be determined positive constants and \(\epsilon\) is small enough. Integrating by parts and using the equation in (54), we obtain

\[
G'(t) = (1 - \alpha) H^{-\alpha}(t) H'(t) + \epsilon \|u_t\|^2 - \epsilon \|Au_t\|^2
\]

\[
- \alpha \epsilon \int_{\Omega} |u_t|^{m-2} u_t u_t \, dx + \beta \epsilon \|u\|_p^p.
\]

To estimate the fourth term of (63), we use Young's inequality,

\[
\int_{\Omega} |u_t|^{m-2} u_t u_t \, dx \leq \int_{\Omega} |u_t|^{m-1} \, dx
\]

\[
\leq \frac{\delta^m}{m} \|u_t\|_m^m + \frac{m - 1}{m} \delta^{-m(m-1)} \|u_t\|_m^m,
\]

where \(\delta\) can be time dependent, since the integral is taken over the \(x\) variable. Take \(\delta\) so that

\[
\delta^{-m(m-1)} = k H^{-\alpha}(t),
\]

for sufficiently large \(k\) to be specified later. Hence together with the definitions of \(E(t)\) and \(H(t)\), it follows that

\[
G'(t) \geq \left[1 - \alpha - \frac{m - 1}{m} \epsilon k \right] H^{-\alpha}(t) H'(t)
\]

\[
+ \epsilon \left(1 + \frac{P}{2}\right) \|u_t\|^2 + \epsilon \left(\frac{P}{2} - 1\right) \|Au_t\|^2
\]

\[
- \frac{\alpha \epsilon k^{1-m}}{m} H^{m-1}(t) \|u_t\|_m^m + \epsilon \|u\|^p H(t) - \epsilon \|u\|^p.
\]

Furthermore, by Lemma 3 we have

\[
\|Au\|^2 \geq \frac{s_1^2 - s_2^2}{s_2^2} \|Au\|^2 + \frac{s_1^2}{s_2^2} \|Au\|^2
\]

\[
\geq \frac{s_1^2}{s_2^2} \|Au\|^2 + s_1^2.
\]

Thus it follows from (66) and (67) that

\[
G'(t) \geq \left[1 - \alpha - \frac{m - 1}{m} \epsilon k \right] H^{-\alpha}(t) H'(t)
\]

\[
+ \epsilon \left(1 + \frac{P}{2}\right) \|u_t\|^2 + \epsilon \left(\frac{P}{2} - 1\right) \frac{s_1^2}{s_2^2} \|Au_t\|^2
\]

\[
- \frac{\alpha \epsilon k^{1-m}}{m} H^{m-1}(t) \|u_t\|_m^m + \epsilon \|u\|^p H(t)
\]

\[
+ \epsilon \left(\frac{P}{2} - 1\right) s_1^2 - \epsilon \|u\|^p.
\]

By taking \(M = E_1\), we obtain

\[
G'(t) \geq \left[1 - \alpha - \frac{m - 1}{m} \epsilon k \right] H^{-\alpha}(t) H'(t)
\]

\[
+ \epsilon \left(1 + \frac{P}{2}\right) \|u_t\|^2 + \epsilon \left(\frac{P}{2} - 1\right) \frac{s_1^2}{s_2^2} \|Au_t\|^2
\]

\[
- \frac{\alpha \epsilon k^{1-m}}{m} H^{m-1}(t) \|u_t\|_m^m + \epsilon \|u\|^p H(t).
\]

Note that, from the definition of \(E(t)\) and (48), we obtain

\[
0 < H(0) \leq H(t) = M - E(t) \leq M - \frac{1}{2} \|Au\|^2 + \frac{b}{p} \|u\|^p
\]

\[
< M - \frac{1}{2} s_1^2 + \frac{b}{p} \|u\|^p \leq \frac{b}{p} \|u\|^p,
\]

\[
(70)
\]

Thus by (70) and the inequality

\[
\|u_t\|_m \leq \|\epsilon\|_p^m,
\]

we obtain

\[
H^{m-1}(t) \|u_t\|_m \leq \left(\frac{b}{p}\right)^{m-1} \|u_t\|_m^{m-1} + \frac{B^2}{\lambda} + 1 \|Au_t\|^2 + \|u\|^p.
\]

(72)

Selecting \(\alpha \in (0, (p - m)/p(m - 1)]\), then together with Lemma 2, we get that

\[
H^{m-1}(t) \|u_t\|_m \leq \left(\frac{b}{p}\right)^{m-1} \left(\frac{B^2}{\lambda} + 1 \|Au_t\|^2 + \|u\|^p \right).\]

(73)
Combining (69) and (73), we have
\[ G'(t) \geq \left[ 1 - \alpha - \frac{m-1}{m} \epsilon k \right] H^{-\alpha}(t) H'(t) 
+ \epsilon \left( 1 + \frac{P}{2} \right) \left\| u_t \right\|^2 + \epsilon \left( \frac{P}{2} - 1 \right) \left( \frac{b^2 s_2 - s_1^2}{2 s_2^2} \right) \left\| \nabla u \right\|^2 
- \frac{a e k^{1-m}}{m} \left( \frac{b}{p} \right)^{a(m-1)} \left( \frac{B^2}{\lambda} + 1 \right) \left\| \nabla u \right\|^2 + \epsilon \left( \frac{p + \epsilon H(t)}{2} \right). \]

From (74) and the equality
\[ H(t) = M + \frac{b}{p} \left\| u \right\|_p^p - \frac{1}{2} \left\| u_t \right\|^2 - \frac{1}{2} \left\| \nabla u \right\|^2, \]
we obtain
\[ G'(t) \geq \left[ 1 - \alpha - \frac{m-1}{m} \epsilon k \right] H^{-\alpha}(t) H'(t) 
+ \epsilon \left( 1 + \frac{P}{2} \right) \left\| u_t \right\|^2 + \epsilon \left( \frac{P}{2} - 1 \right) \left( \frac{b^2 s_2 - s_1^2}{2 s_2^2} \right) \left\| \nabla u \right\|^2 
- \frac{a e k^{1-m}}{m} \left( \frac{b}{p} \right)^{a(m-1)} \left( \frac{B^2}{\lambda} + 1 \right) \left\| \nabla u \right\|^2 + \epsilon \left( \frac{p + \epsilon H(t)}{2} \right) - \frac{a e k^{1-m}}{m} \left( \frac{b}{p} \right)^{a(m-1)} \left( \frac{B^2}{\lambda} + 1 \right) \left\| u \right\|_p^p. \]

Then we can choose k large enough such that the coefficients of \( \left\| \nabla u \right\|^2 \) and \( \left\| u \right\|_p^p \) in (76) are strictly positive, and hence we have
\[ G'(t) \geq \left[ 1 - \alpha - \frac{m-1}{m} \epsilon k \right] H^{-\alpha}(t) H'(t) 
+ \epsilon \beta \left\| u_t \right\|^2 + \epsilon \left\| \nabla u \right\|^2 + \epsilon \left\| u \right\|_p^p + H(t), \]
where \( \beta > 0 \) is the minimum of these coefficients. For fixed \( k \), we pick \( \epsilon \) small enough such that \( 1 - \alpha - ((m-1)/m) \epsilon k \geq 0 \) and
\[ G(0) = H^{1-\alpha}(0) + \epsilon \int_\Omega u_0 u_1 \, dx > 0. \]
Hence
\[ G'(t) \geq \epsilon \beta \left\| u_t \right\|^2 + \epsilon \left\| \nabla u \right\|^2 + \epsilon \left\| u \right\|_p^p + H(t). \]

Thus for \( t \geq 0 \), \( G(t) \) is a nondecreasing function, and we have, for all \( t \geq 0 \),
\[ G(t) \geq G(0) > 0. \] (80)

Here we estimate \( G^{1/(1-\alpha)}(t) \) from above. In the following of this section, we denote by \( C \) the general positive constant which depends on \( |\Omega|, p, \alpha, \) and \( H(0) \).
We use Hölder’s inequality to estimate the term
\[ \left\| \int_\Omega u u_t \, dx \right\| \leq \left\| u \right\| \left\| u_t \right\| \leq C \left\| u \right\|_p \left\| u_t \right\|, \] (81)
which implies
\[ \left\| \int_\Omega u u_t \, dx \right\|^{1/(1-\alpha)} \leq C \left\| u \right\|_p^{1/(1-\alpha)} \left\| u_t \right\|^{1/(1-\alpha)}. \] (82)
Using Young’s inequality
\[ XY \leq \frac{\delta^\mu}{\mu} X^\mu + \frac{\delta^{-\theta}}{\theta^\theta} Y^\theta, \quad X, Y \geq 0, \quad \forall \delta > 0, \quad \frac{1}{\mu} + \frac{1}{\theta} = 1, \] (83)
with \( \delta = 1, \mu = 2(1 - \alpha)/(1 - 2\alpha), \) and \( \theta = 2(1 - \alpha) \), we have
\[ \left\| \int_\Omega u u_t \, dx \right\|^{1/(1-\alpha)} \leq C \left( \left\| u \right\|_p^{2/(1-2\alpha)} + \left\| u_t \right\|^2 \right). \] (84)
By choosing
\[ 0 < \alpha \leq \min \left( \frac{p - 2}{2p}, \frac{p - m}{p(m - 1)} \right), \] (85)
which implies \( 2/(1 - 2\alpha) \leq p \), then using (70) and the inequality
\[ z^\alpha \leq z + 1 \leq \left( 1 + \frac{1}{\alpha} \right) (z + \alpha), \quad \forall z \geq 0, \quad a > 0, \quad 0 < \nu \leq 1, \] (86)
with \( z = \left\| u \right\|_p^p \) and \( a = H(0) \), we deduce
\[ \left\| \int_\Omega u u_t \, dx \right\|^{1/(1-\alpha)} \leq C \left( 1 + \frac{1}{H(0)} \right) \left( \left\| u \right\|_p^p + \left\| u_t \right\|^2 + H(t) \right). \] (87)
Consequently, we have, for all \( t > 0 \),
\[ G^{1/(1-\alpha)}(t) = \left( H^{1-\alpha}(t) + \epsilon \int_\Omega u u_t \, dx \right)^{1/(1-\alpha)} \leq 2^{1/(1-\alpha)} \left( H(t) + \int_\Omega u u_t \, dx \right)^{1/(1-\alpha)} \] (88)
\[ \leq C \left[ H(t) + \left\| u \right\|_p^p + \left\| u_t \right\|^2 \right]. \]
It follows from (79) and (88) that
\[ G'(t) \geq C G^{1/(1-\alpha)}(t). \] (89)
A simple integration of (89) over \((0, \tau)\) then yields
\[
G^{\alpha/(1-\alpha)}(\tau) \geq \frac{1}{G^{\alpha/(1-\alpha)}(0) - \text{Cat} / (1 - \alpha)}.
\] (90)

Therefore (90) shows that \(G(t)\) blows up in a finite time \(T^*\) given by the estimate
\[
T^* \leq \frac{1 - \alpha}{\text{Cat} G^{\alpha/(1-\alpha)}(0)}.
\] (91)

So the solution \(u(x, t)\) blows up in a certain finite time.

**Case 2 \((E(0) < 0)\).** We can take \(M = 0\) in the definition of \(H(t)\); that is,
\[
H(t) = -E(t).
\] (92)

Then we can get our result by the same arguments as in Case 1. This completes the proof of Theorem 4.

**Remark 5.** The earliest blow-up time \(T^*\) can be estimated by
\[
T^* \leq \frac{1 - \alpha}{\text{Cat} G^{\alpha/(1-\alpha)}(0)},
\] (93)

and the larger the \(G(0)\) is, the quicker the blow-up takes place.

**4. Blow-Up Result for \(m = 2\)**

In this section we discuss the blow-up result for the following problem:

\[
\begin{align*}
&u_{tt} + \Delta u - \gamma \Delta u_t + au_t = b |u|^{p-2} u, \\
&\quad x \in \Omega, \quad t \in (0, T), \\
&\frac{\partial u}{\partial \nu} = 0, \quad x \in \Gamma, \quad t \in (0, T), \\
&u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega,
\end{align*}
\] (94)

with the concavity method; see Levine [1, 2].

Denoting the energy of system (94) by
\[
E(t) = \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|\Delta u\|^2 - \frac{b}{p} \|u\|^p
\] (95)

and multiplying the equation of (94) by \(u_t\) and integrating over \(\Omega\), we obtain, for \(t \in [0, T)\),
\[
\frac{d}{dt} E(t) = -\alpha \|u_t\|^2 - \gamma \|\nabla g u_t\|_g^2.
\] (96)

Our result is as follows.

**Theorem 6.** Suppose \(p\) satisfies (10), \(u(x, t)\) is the solution of problem (94), and the initial data \((u_0, u_1)\) satisfies either one of the following conditions:

(i) \(E(0) < 0\);

(ii) \(E(0) = 0\) and \(\int_\Omega u_0 u_1 \, dx > 0\);

(iii) \(0 < E(0) < E_1\) and \(|\mathcal{A}u_0| > s_1\), where
\[
E_1 := \frac{p - 2}{2p} s_1^2, \quad s_1 := \left(\frac{\lambda p/2}{b B p}\right)^{1/(p-2)},
\] (97)

and \(B\) is the embedding constant satisfying
\[
\|u\|_p \leq B \|\Delta u\|.
\] (98)

Then the solution \(u(x, t)\) blows up in finite time \(T\), and
\[
\lim_{t \to T^-} \left(\|u(t)\|^2 + \|\nabla g u_t\|_g^2\right) = +\infty.
\] (99)

**Proof.** Define
\[
L(t) = \|u_t(t)\|^2 + a \int_0^t \|u(s)\|^2 \, ds
\]
\[
+ \gamma \int_0^t \|\nabla g u_t(s)\|_g^2 \, ds + a (T_0 - t) \|u_0\|^2
\]
\[
+ \gamma (T_0 - t) \|\nabla g u_0\|_g^2 + r (t + \tau)^2, \quad t \in [0, T_0],
\] (100)

where \(T_0, r, \text{ and } \tau\) are positive constants which are specified latter.

It is not difficult to see that \(L(t) > 0\) for all \(t \in [0, T_0]\). Furthermore, we have
\[
L'(t) = 2 \int_\Omega uu_t \, dx + a \|u_t\|^2 - a \|u_0\|^2 + \gamma \|\nabla g u_t\|_g^2
\]
\[
- \gamma \|\nabla g u_0\|_g^2 + 2r (t + \tau)
\]
\[
= 2 \int_\Omega uu_t \, dx + 2a \int_0^t \int_\Omega uu_t \, dx \, ds
\]
\[
+ 2\gamma \int_0^t \int_\Omega \left\langle \nabla g u (x, s), \nabla g u_t (x, s) \right\rangle_g \, dx \, ds
\]
\[
+ 2r (t + \tau),
\] (101)

\[
L''(t) = 2 \|u_t\|^2 - 2 \|\mathcal{A}u\|^2 + 2b \|u\|^p + 2r.
\]
Combining (100) and (101), we obtain

\[
LL'' - \frac{P + 2}{4} (L')^2 \\
= 2L [||u_t||^2 - ||Axu||^2 + b ||u||_p^p + r] \\
+ (p + 2) \left\{ ||u||^2 + a \int_0^t ||u(s)||^2 ds \\
+ \gamma \int_0^t \left[ \frac{\partial g}{\partial u} \right]_g^2 ds + r (t + \tau)^2 \right\} \\
\times \left[ ||u||^2 + a \int_0^t ||u_s||^2 ds \\
+ \gamma \int_0^t \left[ \frac{\partial g}{\partial u} \right]_g^2 ds + r (t + \tau)^2 \right] \\
- (p + 2) \left[ ||u||^2 + a \int_0^t ||u(s)||^2 ds \\
+ \gamma \int_0^t \left[ \frac{\partial g}{\partial u} \right]_g^2 ds + r (t + \tau)^2 \right] \\
\times \left[ ||u||^2 + a \int_0^t ||u_s||^2 ds + \gamma \int_0^t \left[ \frac{\partial g}{\partial u} \right]_g^2 ds + r \right].
\]

(102)

It follows from the fact

\[
\left( \int \Omega uu_t \, dx + a \int_0^t \int \Omega uu_s \, dx \, ds \\
+ \gamma \int_0^t \int \Omega \langle \nabla g, u \rangle_\Omega \, dx \, ds + r (t + \tau)^2 \right) \\
\leq \left( ||u||^2 + a \int_0^t ||u(s)||^2 ds \\
+ \gamma \int_0^t \left[ \frac{\partial g}{\partial u} \right]_g^2 ds + r (t + \tau)^2 \right) \\
\times \left( ||u||^2 + a \int_0^t ||u_s||^2 ds \\
+ \gamma \int_0^t \left[ \frac{\partial g}{\partial u} \right]_g^2 ds + r \right)
\]

(103)

that

\[
LL'' - \frac{P + 2}{4} (L')^2 \\
\geq 2L [||u_t||^2 - ||Axu||^2 + b ||u||_p^p + r] \\
- (p + 2) \left[ ||u||^2 + a \int_0^t ||u(s)||^2 ds \\
+ \gamma \int_0^t \left[ \frac{\partial g}{\partial u} \right]_g^2 ds + r (t + \tau)^2 \right] \\
\times \left[ ||u||^2 + a \int_0^t ||u_s||^2 ds + \gamma \int_0^t \left[ \frac{\partial g}{\partial u} \right]_g^2 ds + r \right].
\]

(104)

Noticing that

\[
L(t) \frac{dL''(t)}{dt} - \frac{P + 2}{4} (L'(t))^2 \\
\geq 2L(t) \left[ ||u_t||^2 - ||Axu||^2 + b ||u||_p^p + r \right] \\
+ (p + 2) \left[ a ||u_t||^2 + \gamma \int_0^t \left[ \frac{\partial g}{\partial u} \right]_g^2 ds + \gamma \int_0^t \left[ \frac{\partial g}{\partial u} \right]_g^2 ds + r \right] \\
\times \left[ ||u||^2 + a \int_0^t ||u_s||^2 ds + \gamma \int_0^t \left[ \frac{\partial g}{\partial u} \right]_g^2 ds + r \right]
\]

(105)

then we get that

\[
L(t) \frac{dL''(t)}{dt} - \frac{P + 2}{4} (L'(t))^2 \\
\geq 2L(t) \left[ ||u_t||^2 - ||Axu||^2 + b ||u||_p^p + r \right] \\
+ (p + 2) \left[ a ||u_t||^2 + \gamma \int_0^t \left[ \frac{\partial g}{\partial u} \right]_g^2 ds + \gamma \int_0^t \left[ \frac{\partial g}{\partial u} \right]_g^2 ds + r \right] \\
\times \left[ ||u||^2 + a \int_0^t ||u_s||^2 ds + \gamma \int_0^t \left[ \frac{\partial g}{\partial u} \right]_g^2 ds + r \right]
\]

(106)

where \( Q(t) \) is defined by

\[
Q(t) := -\frac{P + 2}{4} \int_0^t ||u||^2 ds \\
- \frac{a (p + 2)}{2} \int_0^t ||u_t||^2 ds \\
- \frac{\gamma (p + 2)}{2} \int_0^t \left[ \frac{\partial g}{\partial u} \right]_g^2 ds - \frac{P r}{2}.
\]

(107)
By the definition of $E(t)$, we may also write
\[ Q(t) = -pE(t) + \frac{p - 2}{2} \|Au\|^2 - \frac{a(p + 2)}{2} \int_0^t \|u_s\|^2 \, ds - \frac{pr}{2}, \]
and then we have
\[ \lim_{t \to -\infty} \left( \|u(t)\|^2 + \|\nabla u_g(t)\|^2 \right) = +\infty. \] (117)

Case 2 ($E(0) = 0$ and $\int_{\Omega} u_0 u_1 \, dx > 0$). By taking $r = 0$ we have
\[ Q(t) \geq -p \left( E(0) + \frac{r}{2} \right) = 0. \] (118)

Furthermore, we have
\[ L(t) \geq \frac{p + 2}{4} \left( L'(t) \right)^2 \geq 0, \quad t \in [0, T_0]. \] (119)

Denoting $y(t) = L(t)^{-\frac{(p-2)}{4}}$, we obtain that
\[ y''(t) = -\frac{p-2}{4} L(t)^{-\frac{(p+6)}{4}} \left[ L(t) L''(t) - \frac{p+2}{4} (L'(t))^2 \right] \leq 0, \quad t \in [0, T_0]. \] (120)

Since $\int_{\Omega} u_0 u_1 \, dx > 0$, then we have
\[ \|u_0\|^2 > 0, \quad L(0) > 0. \] (121)

Furthermore, we choose suitable positive constants $\tau$ and $T$ such that
\[ 0 < \frac{L(0)}{L'(0)} \leq \frac{p-2}{4} T, \] (122)

and then from the fact that $L'(0) > 0$ we have
\[ L(t) \geq \left( \frac{4L^{(p+2)/4}(0)}{4L(0) - (p-2)L'(0)t} \right)^{4(p-2)}, \] (123)

for some $t > 0$. By the similar arguments as we did in the proof of Case 1, we can prove the desired limit.

Case 3 ($0 < E(0) < E_1$ and $\|Au_0\| > s_1$). It follows from Lemma 3 that there exists $s_2 > s_1$, such that
\[ \|Au\| \geq s_2 > s_1. \] (124)

Thus together with (110) and (124), it follows that
\[ Q(t) \geq -pE(0) + \frac{p-2}{2} (s_1^2 - s_2^2) - \frac{pr}{2}, \]
and then we have
\[ Q(t) \geq \left( -pE(0) + \frac{p-2}{2} (s_1^2 - s_2^2) + \frac{p-2}{2} \right)^{4(p-2)}. \] (125)

By taking $r = ((p-2)/p)(s_2^2 - s_1^2) > 0$, we get
\[ Q(t) \geq 0, \quad L(0) > 0. \] (126)

Furthermore, we have
\[ L(t) \geq \frac{p+2}{4} \left( L'(t) \right)^2 \geq 0, \quad t \in [0, T_0]. \] (127)
Denoting \( y(t) = L(t)^{-(p-2)/4} \), we obtain that
\[
y''(t) = \frac{-2}{4}L(t)^{-(p+6)/4}\left[ L(t)L''(t) - \frac{p+2}{4}(L'(t))^2 \right] 
\leq 0, \quad t \in [0, T].
\]
(128)
Furthermore, we choose suitable positive constants \( \tau \) and \( T \) such that
\[ 0 < \frac{L(0)}{L'(0)} \leq \frac{p-2}{4}T, \]
(129)
and then we have
\[ L(t) \geq \left( \frac{4L^{(p+2)/4}(0)}{4L(0) - (p-2)L'(0)t} \right)^{4(p-2)}, \]
(130)
for some \( t > 0 \). By the similar arguments as we did in the proof of Case 1, we can prove the desired limit. Theorem 5 is established.

### 5. Global Existence Result

In this section we give the result of the global existence of the solution to problem (1) which is similar to [26].

**Theorem 7.** Suppose \( p, m \) satisfy (10)-(11) and \( p \leq m \), the initial data \((u_0, u_1) \in W \times L^2(\Omega)\), and \( u(x, t) \) is the solution of problem (1). Then the solution \( u(x, t) \) is global.

**Proof.** Set
\[ P(t) = \frac{1}{2}||u||^2 + \frac{1}{2}||Au||^2 + \frac{b}{p}||u||_p^p. \]
(131)
After a simple computation, we get that
\[ P'(t) = -a||u||_m^m - \gamma||\nabla g u||_g^2 + 2b\int_{\Omega} |u|^{p-2} uu_t \; dx. \]
(132)
From Young's inequality, it follows that, for any \( \delta > 0 \),
\[ P'(t) \leq -a||u||_m^m - \gamma||\nabla g u||_g^2 + b\delta ||u||_p^p + bC(\delta) ||u||_p^p. \]
(133)
Since \( p \leq m \), then
\[ P'(t) \leq -a||u||_m^m - \gamma||\nabla g u||_g^2 + bC\delta ||u||_p^p + bC(\delta) ||u||_p^p, \]
(134)
where \( C > 0 \) is a constant depending on the domains \( \Omega \) and \( p \) and \( C(\delta) > 0 \) is a constant depending on \( \delta \). Then we get
\[ P'(t) \leq c_1 + c_2 P(t); \]
(135)
and otherwise, if \( ||u||_m \leq 1 \), from (134), we obtain
\[ P'(t) \leq bC\delta + bC(\delta)||u||_p^p \leq bC\delta + bC(\delta) P(t). \]
(137)
By integrating (135) we have, for \( t \geq 0 \),
\[ P(t) \leq (P(0) + c_3)e^{c_4}, \]
(138)
where \( c_3 = c_1/c_2 \). Theorem 7 is established.

### 6. Global Asymptotic Stability Result

In this section we establish the global asymptotic stability result to problem (1) with \( b > 0 \) small enough and \( p = 2 \) or \( b = 0 \) (linear source term or no source term).

**Theorem 8.** Suppose \( m \) satisfies (II), the initial data \((u_0, u_1) \in W \times L^2(\Omega)\), and \( u(x, t) \) be the solution of problem (1) with \( 0 < b < \lambda(B^2) \) and \( p = 2 \) or \( b = 0 \) (\( B \) is the constant such that \( ||u|| \leq \beta||Au|| \)). Then the solution \( u(x, t) \) is asymptotically stable, and the following decay estimate holds for \( t > 1 \) sufficiently large:
\[ ||u||_t^2 + ||Au||^2 + ||\nabla g u||_g^2 \leq Ce^{-Ct}, \quad m = 2, \]
(139)
\[ ||u||_t^2 + ||Au||^2 \leq Ct^{1-(m)/m}, \quad m \in (2, 3), \]
\[ ||u||_t^2 + ||Au||^2 \leq Ct^{-2/m}, \quad m \geq 3, \]
where the generic constant \( C > 0 \) depends on \( \Omega, \gamma, a, b, m \), and the initial data \((u_0, u_1)\).

**Proof.** For \( 0 < b < \lambda(B^2) \) and \( p = 2 \) or \( b = 0 \), using (48) with \( p = 2 \) and (3), we have
\[ E(0) \geq E(t) = \frac{1}{2}||u||^2 + \frac{1}{2}||Au||^2 - \frac{b}{2}||u||^2 \]
\[ \geq \frac{1}{2}||u||^2 + \frac{\left(1 - \frac{bB^2}{2\lambda}\right)}{2\lambda}||Au||^2 \]
(140)
where \( b < \lambda(B^2) \) and \( \mu = 1/2 - bB^2/2\lambda \). Then thanks to (43), we have
\[ E(0) - E(t) = a\int_0^t \int_{\Omega} |u|^m \; dx \; ds + \gamma\int_0^t \int_{\Omega} |\nabla g u|^2 \; dx \; ds. \]
(141)
Combining (140) and (141), we obtain
\[ E(0) \geq a\int_0^t \int_{\Omega} |u|^m \; dx \; ds + \gamma\int_0^t \int_{\Omega} |\nabla g u|^2 \; dx \; ds. \]
(142)
Therefore, we get the following estimate:
\[ \int_0^t \int_{\Omega} |u|^m \; dx \; ds \leq \frac{E(0)}{a}, \]
(143)
\[ \int_0^t \int_{\Omega} |\nabla g u|^2 \; dx \; ds \leq \frac{E(0)}{\gamma}. \]
(144)
Multiplying the equation of (1) by $u$ and integrating over $\Omega$, we obtain
\[
\frac{d}{dt} \int_{\Omega} uu_t \, dx - \|u_t\|^2 + \|\mathcal{A}u\|^2 + \gamma \frac{d}{dt} \left\| \nabla_g u_j \right\|^2 = b\|u\|^2.
\] (145)

Multiplying (145) by a small enough positive constant $\delta$ to be determined and adding to the energy identity
\[
\frac{d}{dt} \left[ \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|\mathcal{A}u\|^2 - \frac{b}{p} \|u\|^2 \right] - a \|u_t\|_m^m - \gamma \left\| \nabla_g u_j \right\|^2,
\] (146)
we get that
\[
\frac{d}{dt} \left[ \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|\mathcal{A}u\|^2 - \frac{b}{2} \|u\|^2 \right] + \delta \int_{\Omega} uu_t \, dx + \delta \gamma \left\| \nabla_g u_j \right\|^2
\] (147)
\[
= -a \|u_t\|_m^m - \gamma \left\| \nabla_g u_j \right\|^2 + \delta \|u_t\|^2 - \delta \|\mathcal{A}u\|^2
\] \[\] - \delta \|\mathcal{A}u\|^2 - \delta b \|u\|^2.
Set
\[
J(t) := \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|\mathcal{A}u\|^2 - \frac{b}{2} \|u\|^2
\] (148)
\[
+ \delta \int_{\Omega} uu_t \, dx + \delta \gamma \left\| \nabla_g u_j \right\|^2.
\]
Then the following inequality holds:
\[
\frac{d}{dt} J(t) = -a \|u_t\|_m^m - \gamma \left\| \nabla_g u_j \right\|^2 + \delta \|u_t\|^2
\] \[\] - \delta \|\mathcal{A}u\|^2 - a \delta \int_{\Omega} |u_t|^{m-2} u_t \, dx + \delta b \|u\|^2
\[
\leq -a \|u_t\|_m^m - \gamma \left\| \nabla_g u_j \right\|^2 + \delta \|u_t\|^2 - \delta \|\mathcal{A}u\|^2
\] \[\] + a \delta \int_{\Omega} |u_t|^{m-1} |u_t| \, dx + \delta b \|u\|^2
\[
= 2 \delta \|u_t\|^2 - a \|u_t\|_m^m - \gamma \left\| \nabla_g u_j \right\|^2 - 2 \delta E(t)
\] \[\] + a \delta \int_{\Omega} |u_t|^{m-1} |u_t| \, dx.
(149)

Integrating (149) with respect to $t$ over $(0, t)$, we obtain
\[
2 \delta \int_0^t E(s) \, ds \leq J(0) - J(t) + 2 \delta \int_0^t \|u_t\|^2 \, ds - a \int_0^t u_t^m \, ds
\] \[\] - \gamma \int_0^t \left\| \nabla_g u_j \right\|^2 \, ds
\] \[\] + a \delta \int_0^t |u_t|^{m-1} |u_t| \, dx \, ds.
(150)
Combining the nondecreasing property of $E(t)$, we get the estimate
\[
2 \delta t E(t) \leq J(0) - J(t) + 2 \delta \int_0^t \|u_t\|^2 \, ds - a \int_0^t u_t^m \, ds
\] \[\] - \gamma \int_0^t \left\| \nabla_g u_j \right\|^2 \, ds + a \delta \int_0^t |u_t|^{m-1} |u_t| \, dx \, ds.
(151)
From Young’s inequality and the definition of $J(t)$ and (140), we have
\[
J(t) \geq \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|\mathcal{A}u\|^2 - \frac{b}{2} \|u\|^2 - \frac{\delta}{2} \|u\|^2
\] \[\] \geq \left( \frac{1}{2} - \frac{\delta}{2} \right) \|u_t\|^2 + \left( \mu - \frac{\delta B^2}{2\lambda} \right) \|\mathcal{A}u\|^2 + \delta \gamma \left\| \nabla_g u_j \right\|^2
\] \[\] \geq \left( \frac{1}{2} - \frac{\delta}{2} \right) \|u_t\|^2 + \left( \mu - \frac{\delta B^2}{2\lambda} \right) \||\mathcal{A}u\|^2 + \delta \gamma \left\| \nabla_g u_j \right\|^2
\] \[\] \geq 0,
(152)
for $\delta$ small enough. Therefore, the following estimate holds:
\[
2 \delta t E(t) \leq J(0) - J(t) + 2 \delta \int_0^t \|u_t\|^2 \, ds + a \delta \int_0^t |u_t|^{m-1} |u_t| \, dx \, ds
\] \[\] =: J(0) + I_1 + I_2.
(153)
Using Hölder’s inequality, (43), (140), and (143), we obtain
\[
I_1 = 2 \delta \int_0^t \|u_t\|^2 \, dx \, ds
\] \[\] \leq 2 \delta \left( \int_0^t \|u_t\|^2 \, dx \, ds \right)^{(m-2)/m} \left( \int_0^t \|u_t\|^m \, dx \, ds \right)^{2/m}
\] \[\] \leq 2 \delta \left( \frac{E(0)}{a} \right)^{2/m} \left( \int_0^t \|u_t\|^m \, dx \, ds \right)^{2/m},
(154)
where $C(\Omega)$ is a positive constant depending on $\Omega$, and

$$I_2 = a\delta \int_0^t \int_\Omega \|u_i\|^{m-1} |u| \, dx \, ds$$

$$\leq a\delta \left( \int_0^t \int_\Omega \|u_i\|^m \, dx \, ds \right)^{(m-1)/m} \left( \int_0^t \int_\Omega |u|^m \, dx \, ds \right)^{1/m}$$

$$\leq a\delta C_* \left( \frac{E(0)}{a} \right)^{(m-1)/m} \left( \frac{E(0)}{\mu} \right)^{1/2} t^{1/m},$$

(155)

where $C_*$ is the constant such that $\|u\|_m \leq C_* \|\mathcal{A}u\|$. It follows from (153)–(155) that

$$2\delta E(t) \leq J(0) t^{-1} + 2\delta \left( \frac{E(0)}{a} \right)^{2/m} (C(\Omega))^{(m-2)/m} t^{-2/m}$$

$$+ a\delta C_* \left( \frac{E(0)}{a} \right)^{(m-1)/m} \left( \frac{E(0)}{\mu} \right)^{1/2} t^{(1-m)/m}.$$  

(156)

Therefore, there are positive constants $K_i (i = 1, 2, 3)$ such that

$$E(t) \leq K_1 t^{-1} + K_2 t^{-2/m} + K_3 t^{(1-m)/m}.$$  

(157)

Hence together with (140), we have, for $t > 1$ sufficiently large, there exist positive constants $C > 0$, such that

$$\|u_i\|^2 + \|\mathcal{A}u\|^2 \leq Ct^{(1-m)/m}, \quad m \in [2, 3),$$

$$\|u_i\|^2 + \|\mathcal{A}u\|^2 \leq Ct^{-2/m}, \quad m \geq 3.$$  

(158)

In particular, in the case $m = 2$, we obtain from (149) with $m = 2$

$$\frac{d}{dt} J(t) + a \|u_i\|^2 + \gamma \|\nabla g u_i\|^2 - \delta \|u_i\|^2$$

$$+ \delta \|\mathcal{A}u\|^2 + a\delta \int_\Omega u_i u \, dx - \delta b \|u\|^2 = 0.$$  

(159)

Multiplying $J(t)$ by a small enough positive constant $\delta^*$ to be determined and adding to (159), we obtain

$$0 = \frac{d}{dt} J(t) + a \|u_i\|^2 + \gamma \|\nabla g u_i\|^2 - \delta \|u_i\|^2$$

$$+ \delta \|\mathcal{A}u\|^2 + a\delta \int_\Omega u_i u \, dx - \delta b \|u\|^2$$

$$+ \delta^* J(t) - \frac{1}{2} \delta^* \|u_i\|^2 - \frac{1}{2} \delta^* \|\mathcal{A}u\|^2 + \frac{b}{2} \delta^* \|u\|^2$$

$$- \delta \delta^* \int_\Omega uu \, dx - \delta \delta^* \gamma \|\nabla g u_i\|^2$$

$$\geq \frac{d}{dt} J(t) + \delta^* J(t) + \left( \frac{a}{2} - \delta - \delta^* \right) \|u_i\|^2$$

$$+ \gamma \|\nabla g u_i\|^2 + \left( \delta - \delta^* \right) \|\mathcal{A}u\|^2 + \frac{\delta^* b}{2} \|u\|^2$$

$$- \left( \delta b + \frac{\delta^2 \delta^*}{2} + \frac{a\delta^2}{2} \right) \|u\|^2 - \delta \delta^* \gamma \|\nabla g u_i\|^2.$$  

(160)

Using (48) with $p = 2$ and (3), we have

$$\|u\|^2 \leq \frac{B^2}{\lambda} \|\mathcal{A}u\|^2.$$  

(161)

It is not difficult to see that

$$\|\nabla g u_i\|^2 \leq C \|\mathcal{A}u\|^2,$$  

(162)

for some positive constant $C$. Together with (160)–(162), we can get the following estimate:

$$0 \geq \frac{d}{dt} J(t) + \delta^* J(t) + \left( \frac{a}{2} - \delta - \delta^* \right) \|u_i\|^2 + \gamma \|\nabla g u_i\|^2$$

$$+ \left[ \delta - \delta^* - \left( \delta b + \frac{\delta^2 \delta^*}{2} + \frac{a\delta^2}{2} \right) \frac{B^2}{\lambda} - \delta \delta^* \gamma C \right] \|\mathcal{A}u\|^2$$

$$+ \delta b \|u\|^2.$$  

(163)

We can choose $\delta$ and $\delta^*$ ($\delta^* < \delta$) sufficiently small with $0 < b < \lambda/B^2$ such that the following estimate holds:

$$\frac{d}{dt} J(t) + \delta^* J(t) \leq 0.$$  

(164)

Then we can get that

$$J(t) \leq J(0) e^{-\delta^* t}.$$  

(165)

Thus due to (152), we have

$$\|u_i\|^2 + \|\mathcal{A}u\|^2 + \|\nabla g u_i\|^2 \leq Ce^{-Ct},$$  

(166)

where the generic constants $C > 0$ depend on $\Omega$, $\gamma$, $a$, $b$, and the initial data $(u_0, u_1)$. This completes the proof of Theorem 8.
Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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