A method of determining effective elastic properties of honeycomb cores based on equal strain energy

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Abstract A computational homogenization technique (CHT) based on the finite element method (FEM) is discussed to predict the effective elastic properties of honeycomb structures. The need of periodic boundary conditions (BCs) is revealed through the analysis for in-plane and out-of-plane shear moduli of models with different cell numbers. After applying periodic BCs on the representative volume element (RVE), comparison between the volume-average stress method and the boundary stress method is performed, and a new method based on the equality of strain energy to obtain all non-zero components of the stiffness tensor is proposed. Results of finite element (FE) analysis show that the volume-average stress and the boundary stress keep a consistency over different cell geometries and forms. The strain energy method obtains values that differ from those of the volume-average method for non-diagonal terms in the stiffness matrix. Analysis has been done on numerical results for thin-wall honeycombs and different geometries of angles between oblique and vertical walls. The inaccuracy of the volume-average method in terms of the strain energy is shown by numerical benchmarks.

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1. Introduction

For the past several decades, sandwich plates with a honeycomb core have been widely used in the field of aviation. In understanding the behavior of sandwich structures under different types of load, the honeycomb core is often regarded as a homogeneous solid with orthotropic elastic properties.1 As a result, research on the effective elastic properties of the honeycomb core is of great essence for the calculation and design of honeycomb sandwich structures.

A computational homogenization technique (CHT) has been found to be a powerful method to predict the effective properties of structures with periodic media. In order to obtain the effective stiffness tensor, which relates to the equivalent strain and stress, this process is divided into solving six elementary boundary value problems, which refer to uniaxial tensile and shear in three directions.2–5 The equivalent strain is determined after applying the unit displacement boundary conditions (BCs) on the representative volume element (RVE) cell corresponding to one of the six elementary problems. Different

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methods have been used when dealing with the equivalent stress. Volume-average stress is often used to determine effective properties in the literature.\textsuperscript{2–6} Catapano and Montemurro\textsuperscript{7} investigated the elastic behavior of a honeycomb with double-thickness vertical walls over a wide range of relative densities and cell geometries. The strain-energy based numerical homogenization technique was also used by Catapano and Jumel\textsuperscript{8} in determining the elastic properties of particulate-polymer composites. Montemurro et al.\textsuperscript{9} performed an optimization procedure at both meso and macro scales to obtain a true global optimum configuration of sandwich panels by using the NURBS curves to describe the shape of the unit cell. Malek and Gibson\textsuperscript{10} got numerical results of a thick-wall honeycomb closer to their analytical solutions by considering nodes at the intersections of vertical and inclined walls. Shi and Tong\textsuperscript{11} focused on the transverse shear stiffness of honeycomb cores by the two-scale method of homogenization for periodic media. Many researchers also seek the stress on the boundary of the RVE cell. Li et al.\textsuperscript{12} used the sum of the node force on the boundary of the RVE cell to obtain the equivalent stress. Papka and Kyriakides\textsuperscript{13} set plates on the top and bottom of the RVE cell to exert BCs. However, regarding honeycomb structures as a combination of cell walls and air, the stress variations on the boundary cause the boundary stress inaccurate to calculate effective properties. Some divergence still exists in numerical results of regular hexagonal honeycomb structures with analytical solutions, especially for the in-plane and out-of-plane shear moduli. From the definitions of effective elastic properties expressed by Yu and Tang,\textsuperscript{14} the equivalent stress is required to make sure that the RVE cell and the corresponding unit volume of the homogeneous solid undergo the same strain energy. Hence, the whole honeycomb structure containing a finite number of RVE cells have the same strain energy as that of the whole volume of the homogeneous solid. The mathematical homogenization theory (MHT) has proven that the strain energy in the RVE can be determined by the volume-average stress and strain.\textsuperscript{15} However, it is not always suitable for the calculation of the volume-average stress method in the CHT. The volume-average method cannot get all precise values in the stiffness matrix, and it is found to get larger strain energy than that obtained from direct analysis in two-dimensional porous composites by Hollister and Kikuchi.\textsuperscript{16} Therefore, we focus on the total strain energy of the RVE cell and propose a new method to determine all the components of the stiffness tensor more accurately in terms of the strain energy.

In Section 2, the differences between the proposed energy method and previous methods are analyzed. A process to obtain 9 components of the effective stiffness tensor based on the energy method is introduced. Then, finite element (FE) models are discussed in Section 3. Convergence analysis has been done over material properties, mesh sizes, and BCs applied on the whole model. In addition, two models are proposed to acquire in-plane and out-of-plane shear moduli according to the different deformations of a single RVE cell and a finite number of RVE cells under the same loading. After establishing appropriate models for honeycomb structures, numerical results over a range of cell geometries are compared to analytical solutions in literature in the next section. Finally, Section 5 ends the paper with some conclusions.

2. Prediction method

2.1. Introduction of a computational homogeneous technique

Previous experimental data and theory have proven that a honeycomb core can be classified as an orthotropic material.\textsuperscript{17} Under this assumption, a honeycomb core conforms to generalized Hook’s law\textsuperscript{18} as

$$\begin{bmatrix}
\bar{\sigma}_{11} \\
\bar{\sigma}_{22} \\
\bar{\sigma}_{33} \\
\bar{\tau}_{12} \\
\bar{\tau}_{23} \\
\bar{\tau}_{13}
\end{bmatrix} =
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\
C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66}
\end{bmatrix}
\begin{bmatrix}
\bar{e}_{11} \\
\bar{e}_{22} \\
\bar{e}_{33} \\
\bar{\gamma}_{12} \\
\bar{\gamma}_{23} \\
\bar{\gamma}_{13}
\end{bmatrix}$$

(1)

where $\bar{\sigma}$ and $\bar{\tau}$ are, respectively, the equivalent stress and strain tensors for the whole geometry of an RVE cell. $C_{ij}$ is one of the components of the stiffness tensor $C$ which is symmetric as $C_{ij} = C_{ji}$. In addition, the shear strain relates the components of the strain tensor as follows,

$$\left\{ \begin{array}{l}
\bar{e}_{11} \rightarrow \bar{e}_{11} = \frac{\partial u}{\partial x}
\\
\bar{e}_{22} \rightarrow \bar{e}_{22} = \frac{\partial v}{\partial y}
\\
\bar{e}_{33} \rightarrow \bar{e}_{33} = \frac{\partial w}{\partial z}
\\
\bar{\gamma}_{12} \rightarrow \bar{\gamma}_{12} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}
\\
\bar{\gamma}_{23} \rightarrow \bar{\gamma}_{23} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}
\\
\bar{\gamma}_{13} \rightarrow \bar{\gamma}_{13} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}
\end{array} \right.$$

(2)

where $u$, $v$, and $w$ represent the displacements in the $x$, $y$, and $z$ directions.

To determine the effective stiffness matrix of the RVE cell, six elementary BCs are applied on the RVE cell, which refer to three uniaxial extensions and three shear deformations. For each load case, only one component of the strain tensor is not zero. Then the relative stiffness component is determined by the equivalent stress. Take $C_{11}$ for example,

$$C_{11} = \frac{\sigma_{11}}{\epsilon_{11}} \quad \text{in load case} \quad \epsilon_{11} \neq 0$$

(3)

After obtaining all the 9 independent components in the stiffness matrix, engineering constants can be derived from the compliance matrix which is the inverse of the stiffness matrix.

2.2. Energy method

Assuming that an elementary shear boundary displacement is applied on the RVE cell ($\bar{\gamma}_{kl} \neq 0$), Eq. (4) is tenable since the boundary of the RVE cell has an identical displacement.

$$\frac{1}{V} \int \gamma_{kl} \delta \bar{v} = \overline{\gamma}_{kl}$$

(4)

where $V$ represents the total volume of the RVE and subscript “$\overline{\gamma}$” stands for the certain BC. The strain energy of the RVE cell under certain loading can be determined by the FE result as

$$U = \frac{1}{2} \int \sigma_{ij} \epsilon_{ij} \delta v \quad i,j = 1, 2, 3$$

(5)
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To guarantee the RVE cell and the corresponding volume of the homogeneous material having the same strain energy, the equivalent stress in this method is

\[
\bar{\sigma}_{ij} = \frac{U}{(1/2)7U7iV} = \frac{\int \sigma_{ij} \delta V}{7U7iV} 
\]

(6)

Hence, the effective modulus through this energy method is

\[
G = \frac{\int \sigma_{ij} \delta V}{7U7iV} \quad i,j = 1,2,3 
\]

(7)

In Section 2.1, a CHT has been introduced which obtains components of the stiffness tensor by solving six elementary BC problems. However, in this energy method expressed in Eq. (7), only one component of the equivalent stress tensor is non-zero, which means that only one component of the equivalent stress tensor can be calculated through Eq. (6) (i.e., only one component of the equivalent stress \( \bar{\sigma}_{ij} \) contributes to the strain energy). Thus, only diagonal components in the stiffness matrix can be acquired by the six elementary BC problems. In order to get all the 9 independent elastic constants of the corresponding homogeneous solid, a bi-axial strain field is applied to obtain the value of \( C_{ij} \) in Eq. (1).

Taking \( C_{12} \) for example, the BC is set as \( \tilde{u}_{11} = \tilde{u}_{22} = \tilde{\tau} \) (two uniaxial tensions applied simultaneously) while the displacements in other directions are zero. According to the equality of the strain energy,

\[
\frac{1}{2} \tilde{\tau}_{11} \tilde{u}_{11} + \frac{1}{2} \tilde{\tau}_{22} \tilde{u}_{22} = U 
\]

(8)

\[
\bar{\sigma}_{11} + \bar{\sigma}_{22} = 2U \frac{\tilde{\tau}}{7U7iV} 
\]

(9)

From Eq. (1),

\[
\bar{\sigma}_{11} = C_{11} \tilde{u}_{11} + C_{12} \tilde{u}_{22} 
\]

(10)

\[
\bar{\sigma}_{22} = C_{22} \tilde{u}_{11} + C_{22} \tilde{u}_{22} 
\]

(11)

Adding Eqs. (10) and (11), and considering the symmetry of the stiffness matrix,

\[
C_{12} = \frac{1}{2} \left( \bar{\sigma}_{11} + \bar{\sigma}_{22} \right) - C_{11} - C_{22} = \frac{1}{2} \left( 2U \frac{\tilde{\tau}}{7U7iV} - C_{11} - C_{22} \right) 
\]

(12)

\( C_{13} \) and \( C_{23} \) can also be acquired by exerting similar BCs on the RVE cell. With the diagonal components obtained by the six elementary load cases, the entire stiffness matrix is determined and 9 engineering constants are then calculated from the compliance matrix.

2.3. Comparative study

As mentioned in Section 1, different methods have been applied to determine the equivalent stress. In this section, a comparative study is done among the volume-average method, the boundary method, and the energy method.

(1) Volume-average method

The equivalent stress in this method is calculated as follows:

\[
\bar{\sigma}_{ij} = \frac{1}{V} \int \sigma_{ij} \delta V 
\]

(13)

where \( \sigma_{ij} \) is the corresponding stress component in the stress field obtained from the FE analysis. The related shear modulus can be written as

\[
G_i = \frac{1}{V} \int \sigma_{ii} \delta v = \frac{2}{V} \int \sigma_{ii} \delta v \cdot \int \delta_{ij} \delta v 
\]

(14)

(2) Boundary stress method

The equivalent stress is obtained by summing up node forces on the boundary as

\[
\bar{\sigma}_{ij} = \frac{\sum F_{ik}}{S} 
\]

(15)

where \( F_{ik} \) is the corresponding node force on the boundary of the RVE cell and \( S \) means the area of the section on which the displacement is applied.

Thus, the effective shear modulus is determined by

\[
G_i = \frac{\sum F_{ik}}{S} 
\]

(16)

(3) Energy method

The effective property obtained by the energy method is shown in Eq. (7).

We review Eqs. (14) and (16) to compare the volume-average method and the boundary stress method. Without loss of generality, the volume-average stress in the RVE cell is

\[
\frac{1}{V} \int \sigma_{ij} \delta v = \frac{1}{V} \int \sigma_{ii} \delta v = \frac{1}{V} \int \sigma_{ii} \delta v \frac{\partial x_j}{\partial x_i} 
\]

(17)

where \( i,j,k \in \{1,2,3\} \).

Eq. (17) shows that the average stress depends uniquely on the surface loading. Here, a further proof is given to show that the average stress only relates to the average stress on the surface where the unit displacement is applied.

For a rectangular RVE cell, as shown in Fig. 1, the six boundary surfaces are named \( A \) to \( F \) respectively.

\[
\int_{\partial V} \sigma_{ij} \delta u d \mathbf{s} = \int_{\partial A} \sigma_{ij} \delta u d \mathbf{s} + \int_{\partial B} \sigma_{ij} \delta u d \mathbf{s} + \int_{\partial C} \sigma_{ij} \delta u d \mathbf{s} + \int_{\partial D} \sigma_{ij} \delta u d \mathbf{s} + \int_{\partial E} \sigma_{ij} \delta u d \mathbf{s} + \int_{\partial F} \sigma_{ij} \delta u d \mathbf{s} 
\]

(18)

Fig. 1  A rectangular RVE.
Consider the six elementary displacement BCs:

(1) $\tau_{11} \neq 0$

In this periodic media, for the two points having the same local load on Surfaces $E$ and $F$,

$$x_{1E} - x_{1F} = (1 + \tau_{11})u_{EF}$$  

where $u_{EF}$ means the height of the RVE.

As a result of the symmetry of both the honeycomb RVE and the applied BCs, the stress also distributes symmetrically, i.e.,

$$\begin{align}
(\sigma_{12})_{E,F} &= 0 \\
(\sigma_{11})_{E,F} &= 0 
\end{align}$$

Considering the correspondence between Surfaces $E$ and $F$ in the periodicity of the RVE, Eq. (18) can be written as

$$\int_{E} \sigma_{1i} n_{i} dS = \int \left( \sigma_{11} n_{1} + \sigma_{12} n_{2} \right) dS = \int \sigma_{11} n_{1} dS$$

Then under this BC, the volume-average stress can be determined from Eqs. (17) and (21) as

$$\frac{1}{V} \int_{V} \sigma_{1i} dV = \frac{u_{EF}}{V} \int_{V} \sigma_{11}(1 + \tau_{11}) u_{EF} dS$$

Further results can be acquired for uniaxial tensions in other two directions ($\tau_{22} \neq 0$ and $\tau_{33} \neq 0$).

(2) $\tau_{12} \neq 0$

In this periodic media, for the two points having the same local load on Surfaces $E$ and $F$,

$$x_{2E} - x_{2F} = (1 + \tau_{12})u_{EF}$$

As a result of the symmetry of both the honeycomb RVE and the applied BCs, the stress also distributes symmetrically, i.e.,

$$\begin{align}
(\sigma_{11})_{E,F} &= 0 \\
(\sigma_{22})_{E,F} &= 0 
\end{align}$$

Considering the correspondence between Surfaces $E$ and $F$ in the periodicity of the RVE, Eq. (18) can be written as

$$\int_{F} \left( \sigma_{12} n_{1} + \sigma_{22} n_{2} \right) dS = 0$$

Then under this BC, the volume-average stress can be determined from Eqs. (17) and (25) as

$$\frac{1}{V} \int_{V} \sigma_{12} dV = \frac{u_{EF}}{V} \int_{V} \sigma_{12}(1 + \tau_{12}) u_{EF} dS$$

Similar results can be acquired for shear deformations in other two directions ($\tau_{13} \neq 0$ and $\tau_{23} \neq 0$).

The above analysis shows that under both the tensions and shear deformations, the volume-average stress only depends on the average stress on the surface where the unit displacement is applied. Eqs. (22) and (26) show an equality of the volume-average stress and the boundary stress, which will be discussed later in Section 4.

We review Eqs. (7) and (14) to compare the energy method and the volume-average method. The difference lies in $\int \sigma_{ij} dV \cdot \int \varepsilon_{ij} dV$ and $2V \int \sigma_{ij} \varepsilon_{ij} dV$.

Without loss of generality, for the energy method, we have

$$\int_{V} \sigma_{ij} \varepsilon_{ij} dV = \int_{V} \sigma_{ij} \varepsilon_{ij} dS = \int_{V} \sigma_{ij} \varepsilon_{ij} dV$$

where $i, j, k \in \{1, 2, 3\}$.

Eq. (27) indicates an equality between the volume-average method and the energy method when calculating the components in the stiffness matrix. However, as mentioned in Section 2.2 for the operating process of the energy method, the equilibrium shown in Eq. (27) only exists for the diagonal elements in the stiffness matrix, because non-diagonal components can’t be calculated directly by Eq. (7). In other words, the volume-average method in the calculation of the equivalent stress is only acceptable for diagonal elements like $C_{11}$, $C_{22}$, and so on, while divergence exists on non-diagonal elements.

Taking the calculation of $C_{12}$ for example, in the volume-average method, $C_{12}$ is calculated as

$$C_{12} = \frac{1}{V^2} \int \sigma_{1i} dV$$

where $\sigma_{1i}$ stands for the local stress when $\tau_{22} = \tau_{33} = \tau$ is applied.

Eq. (12) shows the calculation in the energy method, which considers the total strain energy when applied to $\tau_{11} = \tau_{22} = \tau$.

On one hand, by using $C_{12}$ acquired by the volume-average method as Eq. (28), the total strain energy of the RVE under bi-axial BCs $\tau_{11} = \tau_{22} = \tau$ is written as

$$U' = \frac{1}{2} V(\sigma_{11} \varepsilon_{11} + \sigma_{22} \varepsilon_{22}) = \frac{V^2}{2} (C_{11} + C_{22} + 2C_{12})$$

where $\sigma_{1i}$ means the stress field inside the RVE when a mono-axial strain $\varepsilon_{22} = \varepsilon_{33} = \tau$ is employed, which is the process of the volume-average method. On the other hand, similar to the problem for the work and energy under several loads (which is often used for the introduction of Maxwell’s reciprocal theorem), the total strain energy can be written as

$$U = \frac{V^2}{2} (C_{11} + C_{22}) + \int \sigma_{1i} \varepsilon_{1i} dV$$

In this equation, $\sigma_{1i}$ also means the stress field inside the RVE when a mono-axial strain $\tau_{22} = \tau_{33} = \tau$ is employed.

Comparison between Eqs. (29) and (30) shows that divergence lies in $\tau \int \sigma_{1i} \varepsilon_{1i} dV$ and $\int \sigma_{1i} \varepsilon_{1i} dV$.

Set $d_0 = \int \sigma_{1i} dV \cdot \int \varepsilon_{1i} dV$ and $d_2 = \int V \sigma_{1i} \varepsilon_{1i} dV$.

For a homogeneous material (with a stiffness component $C_{12}$), $d_1 = C_{12} \int (\varepsilon_{12} + \varepsilon_{13}) dV$ and $d_2 = C_{12} \int \varepsilon_{11} dV$.

In FE software ABAQUS, strain and stress fields are present in every integral point inside a single element, which
form the entire mesh. Therefore, the integrals of \( d_1 \) and \( d_2 \) transform into the summation of each element. Assuming that in the FE model, the corresponding strain in each element is \( \varepsilon_{ij}^{x y} \), and the volume in each element as \( V \), then

\[
d_1 = C_{12} (x_1 y_1 z_1 + x_2 y_2 z_2 + \ldots + x_n y_n z_n) (y_1 z_1 + y_2 z_2 + \ldots + y_n z_n)
\]

\[
d_2 = C_{12} (x_1 y_1 z_1 + x_2 y_2 z_2 + \ldots + x_n y_n z_n) (z_1 + z_2 + \ldots + z_n)
\]

Comparison between Eqs. (31) and (32) shows that the inhomogeneity of the strain distribution is the main reason that causes the difference between the volume-average method and the energy method. Numerical results indicate that a relatively greater strain exists in the air zone than that in the wall, and the impact of the variation of the strain within the wall material is insignificant mainly due to its small volume fraction.

According to all the above analysis, the difference of the acquired effective properties between the volume-average method and the energy method, especially for the in-plane moduli, depends on the volume fraction of the air zone and the inhomogeneity of the strain field in the RVE cell. Moreover, all these factors mentioned relate to the wall thickness and cell geometries.

![Representative volume element (RVE) in whole structure](image1)

![Size of RVE](image2)

![FE model of RVE](image3)

**Fig. 2** Single-wall-thickness honeycomb structure.

### 3. FE model

#### 3.1. Convergence

The RVE cell chosen for a single-wall-thickness honeycomb structure is shown in Fig. 2. As the vertical and oblique walls have the same thickness in the single-wall-thickness honeycomb, the cell geometric parameters in Fig. 2(b) can be \( t_1 = t \), \( t_2 = t/2 \). Three parameters including \( l/t \), \( h/t \) and \( \theta \) determine the geometric configuration of the RVE cell. In later analysis, we focus on the most commonly used honeycombs, whose \( l \) and \( h \) remain the same as \( l = h = 15 \). This absolute value is not significant as geometries only provide non-dimensional coefficients for the effective properties. Moreover, the core height is set to \( h_c = 50 \) and it is in the range where the core height has little influence on effective properties according to Ref. 2.

Regarding the honeycomb structure as a two-phase mixture of the core material and air as shown in Fig. 2(c), aluminum alloy is the common material for honeycombs, whose modulus \( E = 70 \) GPa and Poisson’s ratio \( v = 0.3 \). The so-called “elastic air” is endowed to the air zone to get the strain field in the air whose modulus and Poisson’s ratio are set as \( E_{\text{air}} = 0.001 \) MPa and \( v_{\text{air}} = 0 \). In order to evaluate whether the elastic constant of elastic air is appropriate, the in-plane shear modulus is chosen owing to its relatively small value compared to other properties.

Fig. 3(a) presents different values of the calculated in-plane shear modulus at different properties of elastic air, in which \( G_c \) is the converged value. The deviation of the shear modulus with \( E_{\text{air}} = 1 \) MPa from the shear modulus with \( E_{\text{air}} = 10^{-2} \) MPa is about 0.8%, while the deviation of \( E_{\text{air}} = 10^{-3} \) MPa from \( 10^{-4} \) MPa is less than 0.001%. This
shows that a smaller modulus of elastic air does not lead to a
significant change of the numerical result. Therefore, it is
appropriate to choose 10^{-3} \text{ MPa} as the elastic property of
the air zone.

Bending deformation dominates in the honeycomb struc-
ture under in-plane loading.\textsuperscript{16} The contributions of axial and
shear deformations are also included for the analysis of the
out-of-plane properties,\textsuperscript{17} so a model composed of 3D solid
elements is established to take into account all the three-
dimensional deformations to get strain and stress fields that
are more precise. As a result of the hourglass phenomenon
existing in the reduced-integration linear element C3D8R (with
enhanced hourglass control) in ABAQUS,\textsuperscript{18} the number of
mesh divisions along the cell wall thickness is studied in the
bending problem, as shown in Fig. 3(b). When the number
of divisions $n = 1$, the numerical result of the effective shear
modulus shows the hourglass phenomenon as the “zero-
energy mode” which makes the stress field in the wall material
almost zero, thus resulting in a very small effective shear mod-
ulus obtained by this mesh size. As the number of divisions
increases, hourglass is suppressed and a converged modulus
is approached. While the deviation between the results of
$n = 5$ and $n = 7$ is less than 1\%, $n = 5$ is chosen for the num-
ber of divisions along the wall thickness.

According to Eq. (3), elementary BCs will be applied to get
an equivalent strain for the whole RVE cell. The essence of this
computational homogeneous method is that the stiffness of the
honeycomb structure is replaced by the stiffness of equivalent
solids. Therefore, it is of great significance to simulate the
defor- mation of the whole structure accurately.

Hence, nonlinear effects brought by the possible large
defor-mation of the wall are taken into account. Based on the
above analysis, numerical results considering nonlinear effects
at varying strain are shown in Fig. 3(c). The results indicate
that a nonlinear effect happens at a strain of 10^{-1}, whose effec-
tive modulus is above 50\% higher than that at a 10^{-3} strain.

The difference between results at 10^{-2} and 10^{-3} strains is
within 1\% as well as the difference between those at 10^{-3}
and 10^{-4} strains. Thus, a strain of 10^{-3} is the choice
value for the elementary BCs. In the meanwhile, the maximum stress
of the honeycomb wall under a 10^{-3} strain BC is 21.1 MPa
from the FE results, which is lower than the yield stress of
5052 aluminum alloy commonly used in commercial honey-
comb structures.\textsuperscript{19,20} Therefore, the honeycomb wall stays in
the linear elastic stage under this loading, and it is appropriate
to use this model to calculate the elastic properties.

\section*{3.2. Boundary conditions}

Catapano and Montemurro\textsuperscript{2} gave detailed BCs when taking
into account the symmetries of the unit cell. As discussed by
Hori and Nemat-Nasser,\textsuperscript{21} the homogeneous stress BC and the
homogeneous strain BC only provide the lower and upper
bounds of effective moduli. The plane-remains-plane homoge-
neous BCs (or unit-displacement BCs) not only over-constrain
the boundaries, but also violate the stress periodicity condi-
tions. In theoretical analysis of the FE model proposed by
Catapano and Montemurro\textsuperscript{2} in the calculation for the in-plane
shear modulus $G_{12}$ and the out-of-plane shear modulus $G_{13}$,
we found that the calculated value differs from different cell
numbers.

\subsection*{3.2.1. $G_{12}$ under unit-displacement BCs}

In this part, the accuracy of the RVE cell selected in Fig. 2(b)
to simulate the entire honeycomb structure under in-plane
shear loading is discussed. The effect of cell numbers on the
effective shear modulus is analyzed, based on which a new
model (The new model is called Model 2, as the model shown
in Fig. 2(c) is called Model 1) is proposed to get more precise
properties of in-plane shear under unit-displacement BCs.

When the honeycomb structure is subject to a shear loading
in the 1-2 direction as shown in Fig. 4, as mentioned before,
bending deformation of the walls dominates in the honeycomb
core. The deflection of the vertical walls and the rotation of the
oblique walls together cause a deformation for the whole hon-
eycomb core under 1-2 shear loading. From the previous the-
dory in literature,\textsuperscript{22} the proportion of the deflection of the
vertical walls in the whole shear deformation on the boundary is

\begin{equation}
\frac{F_{h}k^{2}}{24EI} + \frac{F_{b}k^{2}}{48EI} \sin \theta = 84.2\%.
\end{equation}

Therefore, the deflection of the vertical walls plays an
important role in determining the effective in-plane shear
modulus.

Fig. 4(a) and (b) illustrate the difference between a single
RVE cell and a finite number of RVE cells under the same
in-plane shear loading, which show that the local load on the
vertical walls is not the same, thus causing different deflections.
For a single RVE cell, the deflection of the vertical Wall $AE$ $(CF)$ is

\begin{figure}[h!]
\centering
\includegraphics[width=\textwidth]{Fig_3.png}
\caption{Convergence analysis of FE model.}
\end{figure}
While for the honeycomb core consisting of a finite number of RVE cells, the deflection of Wall $AE$ ($CF$) is

$$w_2 = \frac{Fh^3}{2Ebt^2}$$

(35)

This result has the same value of the deflection as in a finite number of RVE cells.

FE models containing different numbers of cells as respectively $n = 1, 2, 4, 8, 14, 16$ are established to evaluate the effect of the cell number on the effective in-plane shear modulus, and these models have the same BCs as those in Ref. $^2$.

Calculations have been conducted on honeycombs of three geometries:

1. $t = 0.2, \theta = 30^\circ, l = h = 15$
2. $t = 0.4, \theta = 30^\circ, l = h = 15$
3. $t = 0.2, \theta = 45^\circ, l = h = 15$

The FE results in Fig. 5 indicate that for all the geometric situations, the effective in-plane shear modulus increases as more cells are included in the FE models, which is consistent with the previous analysis. Moreover, all the curves have a tendency of approaching a converged value that is closer to the modulus obtained by Model 2. Therefore, the results presented in Fig. 5 can be a validation for the accuracy of Model 2 to simulate a real honeycomb core under in-plane shear loading.

3.2.2. $G_{13}$ under unit-displacement BCs

Similar to Section 3.2.1, the effect of the cell number on the effective out-of-plane shear modulus is analyzed, on the basis of which another new model called Model 3 is proposed to get more precise properties of the out-of-plane shear modulus in the 1-3 direction for the whole honeycomb core.

When the honeycomb structure is subject to a shear loading in the 1-3 direction as shown in Fig. 6, the overall deformation of the honeycomb core is governed by the shear deformations of the vertical and oblique walls. We conduct an analysis on the shear flows inside each wall in a similar way used by Kelsey et al. $^{23}$

$$w_1' = \frac{(F/2)h^3}{2Eb(0.795t)^2} = w_2$$

(36)
566 For the single RVE cell presented in Fig. 6(a) under 1-3 shear loading (with a shear stress $\tau$), the following equation is acquired by equilibrium conditions:

\[
\begin{align*}
q_b &= q_d \\
q_e &= q_e \\
q_a + q_c &= q_b \\
\tau (h + l \sin \theta) \cdot 2l \cos \theta &= (q_b + q_e) l \cos \theta \\
q_a \cdot \frac{1}{2} + (q_b - q_e) l \sin \theta + (q_d - q_c) \cdot \frac{1}{2} &= 0
\end{align*}
\]  

(37)

From Eq. (37),

\[
\begin{align*}
q_a &= 0 \\
q_b &= q_e = q_d = q_c = \tau (h + l \sin \theta)
\end{align*}
\]  

(38)

For the real honeycomb core consisting of a finite number of RVE cells, under the consideration of periodicity,

\[
q_a = q_d = q_e
\]  

(39)

From equilibrium conditions,

\[
\begin{align*}
q_b &= q_e + q_d \\
q_a + q_c &= q_b \\
\tau (h + l \sin \theta) \cdot 2l \cos \theta &= (q_b + q_e) l \cos \theta \\
q_a \cdot \frac{1}{2} + (q_b - q_e) l \sin \theta + (q_d + q_c) \cdot \frac{1}{2} &= 0
\end{align*}
\]  

(40)

From Eqs. (39) and (40), we get

\[
\begin{align*}
q_a &= q_d = q_e = 0 \\
q_b &= q_c = \tau (h + l \sin \theta)
\end{align*}
\]  

(41)

Comparison between Eqs. (38) and (41) shows that under the same shear loading in the 1-3 direction, the shear flow in Wall AE as well as CF varies in a single RVE cell and a finite number of RVE cells. In a single RVE cell, similar to the situation in the simulation of in-plane shear loading, Walls AE and CF suffer higher shear flows than those in a finite number of RVE cells, leading to differences in the deformation for the entire honeycomb structure. The higher shear flows $q_d$ and $q_e$ cause the in-plane bending deformation of the oblique walls as shown in Fig. 7 from the FE analysis of a single RVE cell. It can be seen that the extra bending deformation of the oblique walls is in a direction contributing to the 1-3 shear deformation, and FE results illustrate that this bending deformation is weakening as the cell number increases. For these reasons, a single RVE cell has a larger deformation than that of a finite number of RVE cells under the same loading owing to the bending deformation of the oblique walls, thus having a lower effective shear modulus in the 1-3 direction than that of a finite number of RVE cells.

In order to find an appropriate RVE model to simulate the deformation of the real honeycomb structure under the 1-3 direction shearing precisely, the thickness of the oblique walls is adjusted to suppress the extra bending deformation. The geometry of the proposed new model called Model 3 is determined according to the FE results shown in Fig. 8.

Fig. 8 shows the effective shear modulus along the 1-3 direction of a RVE cell with a geometry of $t = 0.2$, $\theta = 30^\circ$, and $l = h = 15$ at different thicknesses of the oblique walls.
The horizontal dotted line in Fig. 8 represents the converged value of the finite number of RVE cells. It can be seen that when the thickness of the oblique walls \( t_1 = 0.243 = 1.215 \)r, we get a value very close to the converged value for \( G_{13} \). According to these results, \( t_1 = 1.215 \)r is chosen as the geometric change of Model 3 from Model 1, and the accuracy of Model 3 is going to be evaluated later in Fig. 9.

Similar to the in-plane shear loading, FE models containing different numbers of cells as respectively \( n = 1, 2, 4, 8, 16 \) are also used to evaluate the effect of the cell number on the effective out-of-plane shear modulus along the 1-3 direction, and honeycombs of three geometries are also taken into consideration, as shown in Fig. 9. With the number of cells increasing, the effective shear modulus \( G_{13} \) grows and approaches a converged value, as analyzed before. In addition, the converged value is very close to the result of Model 3 for all three cell geometries. Therefore, this new model can provide a relatively more accurate value of \( G_{13} \) than that of Model 1 under this shear loading.

3.2.3. Periodic boundary conditions

As stated before, a single RVE cell cannot provide accurate \( G_{12} \) and \( G_{13} \) as those of the whole honeycomb core with unit-displacement BCs. However, when applied to periodic BCs, the effective properties remain constant with the cell number changing. Moreover, periodic BCs are required in determining the elastic properties of periodic media to ensure the periodicity of displacements and tractions on the boundary of the RVE. In order to generate a symmetrical mesh for the convenience of prescribing the periodic BCs, a whole unit cell is remained for FE analysis.

![In-plane bending deformation of oblique walls under shear loading along 1-3 direction.](image)

**Fig. 7** In-plane bending deformation of oblique walls under shear loading along 1-3 direction.

![Effective shear modulus along 1-3 direction vs thickness of oblique wall AB (BC).](image)

**Fig. 8** Effective shear modulus along 1-3 direction vs thickness of oblique wall AB (BC).

Mathematical expressions can be found in the literature by Whitcomb,\(^{24}\) Xia,\(^{22}\) Li and Wongsto,\(^{26}\) which have been used in periodic media like unidirectional composites and plane and satin weave composites. Constraint equations (CEs) are utilized for periodic BCs of the rectangular solid RVE shown in Fig. 1. These equations can be sorted into three categories, i.e., equations for surfaces, edges, and vertices.

(1) Equations for surfaces

Under three uniaxial tensions and shear deformations \((\varepsilon_{11}^{0}, \varepsilon_{22}^{0}, \varepsilon_{33}^{0}, \gamma_{12}^{0}, \gamma_{13}^{0}, \gamma_{23}^{0})\), CEs can be applied to three pairs of surfaces.

For surfaces perpendicular to 1-axis, i.e., Surfaces \( A \) and \( B \):

\[
\begin{align*}
\varepsilon_{11} &= \varepsilon_{22} = \frac{v_{12}}{\gamma_{12}} W_1 \\
v_{11} &= v_{22} = 0 \\
w_{13} &= w_{32} = 0
\end{align*}
\]

(42)

For surfaces perpendicular to 2-axis, i.e., Surfaces \( A \) and \( B \):

\[
\begin{align*}
\varepsilon_{11} &= \varepsilon_{33} = \frac{v_{13}}{\gamma_{13}} W_3 \\
v_{11} &= v_{33} = 0 \\
w_{12} &= w_{23} = 0
\end{align*}
\]

(43)

For surfaces perpendicular to 3-axis, i.e., Surfaces \( C \) and \( D \):

\[
\begin{align*}
\varepsilon_{22} &= \varepsilon_{33} = \frac{v_{23}}{\gamma_{23}} W_3 \\
v_{22} &= v_{33} = 0 \\
w_{12} &= w_{13} = 0
\end{align*}
\]

(44)

(2) Equations for edges

Two or three in Eqs. (42)(44) are satisfied for nodes on the edges of the RVE. As these constraints are not independent, FE analysis cannot function properly if the CEs for edges are not considered separately as well as the CEs for vertices.

For edges parallel to the 1-axis, i.e., Lines \( h \), \( d \), \( f \) and \( g \): \n
\[
\begin{align*}
u_{10} - u_{2d} &= \frac{v_{11}}{\gamma_{11}} W_1 \\
v_{12} &= v_{2d} = 0 \\
w_{10} &= w_{2d} = 0
\end{align*}
\]

(45)

(46)

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E. G. 726

Similar equations can be given for edges parallel to the 2-axis and 3-axis.

(3) Equations for vertices

Point $d$ is fixed to avoid rigid-body motions of the RVE. Then seven CEs are defined for other vertices with the reference of Point $d$. Equations for Points $f$ and $g$ are given here as examples:

\[
\begin{align*}
\eta_f - \eta_d &= \frac{\rho}{\ell_{12}} W_1 + \frac{\rho}{\ell_{14}} W_2 + \frac{\rho}{\ell_{14}} W_3 \\
\nu_f - \nu_d &= \frac{\rho}{\ell_{12}} W_2 + \frac{\rho}{\ell_{13}} W_3 \\
f_f - w_d &= \frac{\rho}{\ell_{13}} W_3
\end{align*}
\]

Periodic BCs are prescribed in the ABAQUS software combined with Python scripts. Python scripts are used to find corresponding nodes in the mesh according to Eqs. (42)-(49), and submit jobs in the ABAQUS environment. After jobs are finished, post-processing Python scripts are executed to calculate effective moduli by both the volume-average method and the energy method.

4. Results

4.1. Volume-average stress and boundary stress

A total of six cases as listed in Table 1 are analyzed. Controlling parameters include the form of the cell (single-wall thickness and double-wall thickness), the thickness of the wall ($t = 0.2$ and $t = 0.4$), and the angle between vertical and oblique walls ($\theta = 30^\circ$ and $\theta = 45^\circ$). The reason for choosing these parameters is that later analysis has shown that the energy method and the volume-average method have larger divergences under the chosen geometries. For each case, the volume-average stress and the boundary stress are both used to get the effective modulus. $E_1/E'_1$ in Table 1 means the ratio of the modulus obtained by the boundary stress to that by the volume-average stress.

An inspection of Table 1 shows that the values of $E/E'$, $G/G'$, and $t/h$ in all directions are very close to unity for all cases. Therefore, it can be concluded that the volume-average stress is equal to the boundary stress regardless of changes in cell geometries and forms.

As discussed in Section 2.3, Eqs. (22) and (24) show an equality of the volume-average stress and the boundary stress, which is validated by the results in Table 1. Understanding such an equality, only the energy method and the volume-average method are operated in later FE analysis.

4.2. Results by energy method

In this part, we have compared the numerical results by the volume-average method and the energy method. Different cell geometries such as the wall thickness and the angle between vertical and oblique walls are taken into consideration to evaluate the supposed discrepancy between the two methods.

Fig. 10 shows the numerical results of the energy method and the volume-average method at different wall thicknesses which range from 0.2 to 1.0 as the RVE cell has a dimension of $l = h = 15$.

It can be seen in Fig. 10 that the three shear moduli $G_{12}$, $G_{13}$, and $G_{23}$ calculated by the volume-average method and the energy method are all nearly the same with a maximum difference of less than 3%. Since the shear moduli only relate to the diagonal elements in the stiffness matrix while calculated from the compliance matrix, this equilibrium of these three shear moduli can be a validation for Eq. (27). Nevertheless, for the elastic properties relate to non-diagonal components (i.e. $E_1$, $E_2$, $E_3$, $\nu_{12}$, $\nu_{13}$, and $\nu_{23}$), divergences exist between the volume-average method and the energy method.

Fig. 10(d) shows that the volume-average method gets the same results as those of Malek & Gibson’s model, which validates our proper use of the volume-average method.

As stated in Section 2.3, the main motivation that we put forward this energy method is the supposed discrepancy in the calculation of $C_{12}$. It has been proven in Section 2.3 that an inaccurate calculation of $C_{12}$ in the volume-average method leads to an inaccurate strain energy under bi-axial BCs. The discrepancy of the strain energy in each model of unit cells is presented in Fig. 11. It can be seen that the discrepancy remains $1.3$–$1.6\%$ within our computing range. Although the relative error remains nearly constant, the absolute value increases as the wall thickness increases.

For the in-plane elastic properties $E_1$ and $E_2$, within the range of calculation, the absolute value of the discrepancy between the two methods varies with the cell wall thickness increasing. However, in these geometries, as the wall thickness increases, the relative error between these two methods is slightly changed from nearly 8% to 6%. This follows from

<table>
<thead>
<tr>
<th>$t$</th>
<th>$\theta$ (°)</th>
<th>Feature</th>
<th>$E/E'$</th>
<th>$G/G'$</th>
<th>$t/h$</th>
<th>$E_{12}$</th>
<th>$E_{13}$</th>
<th>$E_{23}$</th>
<th>$G_{12}$</th>
<th>$G_{13}$</th>
<th>$G_{23}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>30</td>
<td>Single-wall</td>
<td>1</td>
<td>1.001</td>
<td>1</td>
<td>0.999</td>
<td>1</td>
<td>1</td>
<td>0.999</td>
<td>0.998</td>
<td>0.998</td>
</tr>
<tr>
<td>0.4</td>
<td>30</td>
<td>Single-wall</td>
<td>0.998</td>
<td>1.001</td>
<td>1</td>
<td>0.997</td>
<td>0.999</td>
<td>0.999</td>
<td>0.999</td>
<td>1.002</td>
<td>0.998</td>
</tr>
<tr>
<td>0.2</td>
<td>45</td>
<td>Single-wall</td>
<td>1.001</td>
<td>0.999</td>
<td>0.998</td>
<td>0.996</td>
<td>1</td>
<td>1</td>
<td>1.002</td>
<td>0.998</td>
<td>1</td>
</tr>
<tr>
<td>0.2</td>
<td>30</td>
<td>Double-wall</td>
<td>1</td>
<td>1.001</td>
<td>1</td>
<td>1.002</td>
<td>0.999</td>
<td>0.999</td>
<td>0.999</td>
<td>1.002</td>
<td>1.001</td>
</tr>
<tr>
<td>0.4</td>
<td>30</td>
<td>Double-wall</td>
<td>1.001</td>
<td>1</td>
<td>1</td>
<td>1.001</td>
<td>0.999</td>
<td>0.999</td>
<td>0.999</td>
<td>1.002</td>
<td>1.001</td>
</tr>
<tr>
<td>0.2</td>
<td>45</td>
<td>Double-wall</td>
<td>1</td>
<td>1.001</td>
<td>0.999</td>
<td>1.002</td>
<td>1</td>
<td>1</td>
<td>1.001</td>
<td>0.997</td>
<td>0.997</td>
</tr>
</tbody>
</table>
the fact that the discrepancy of the strain energy under bi-axial BCs remains almost the same as shown in Fig. 11, representing the difference of non-diagonal elements in the stiffness matrix. The numerical results for $E_{3}$ have little discrepancy due to the homogeneous strain and stress fields of the whole honeycomb structure under uniaxial tensions in the 3 directions. As shown in Eqs. (31) and (32), uniform strain and stress fields do not lead to a difference between $d_{1}$ and $d_{2}$. The curves of $v_{13}$ and $v_{23}$ show a similar tendency to those of the in-plane moduli $E_{1}$ and $E_{2}$, which conforms to the hypothesis proposed by Malek and Gibson as $v_{13} = v_{23} = v$ where $v$ is the Poisson’s ratio of the cell wall material.

Fig. 12 shows the numerical results of the energy method and the volume-average method with analytical solutions at different angles between vertical and oblique walls. Cell geometries of three angles $\theta = 30^\circ, 45^\circ, 60^\circ$ and three cell wall thicknesses $t = 0.2, 0.3, 0.4$ are conducted in calculation.

Similar conclusions can be drawn from results illustrated in Fig. 12 with those in Fig. 10. On one hand, approximately the same values for three shear moduli are acquired by the volume-average and energy methods. On the other hand, divergence exists for other properties related to non-diagonal components of the stiffness matrix and this divergence varies with different cell geometries.

Results in Fig. 12 show that the deviation of effective properties obtained by both the volume-average method and the energy method varies as the angle changes from $30^\circ$ to $60^\circ$. Nevertheless, effective elastic properties obtained using the energy method remain at values that are lower than those using the volume-average method. According to the experiment data of a regular honeycomb ($h/l = 1$) provided in

Fig. 10  Numerical results of strain energy method and volume-average stress method vs wall thickness.

Fig. 11  Discrepancy of strain energy of honeycomb cores under bi-axial BCs.
Ref. 2 which are also slightly lower than their numerical results (based on the volume-average method), results by the energy method may provide more accurate values. In a consistency with Fig. 10, numerical results at angles of 45° and 60° also indicate a slight increase of the discrepancy between the two methods as the wall thickness increases.

From the results in Figs. 10–12, it can be concluded that the energy method gets different values of effective properties of a thin-wall thickness honeycomb from those of the volume-average method regardless of the variation of the angle between vertical and oblique walls. It can be proven that the discrepancy of the strain energy under bi-axial BCs can be considered as a symbol for the difference of the two methods in determining effective properties. The inaccuracy in calculating the strain energy shows the inaccurate moduli acquired by the volume-average method.

4.3. Comments on discrepancy

As shown in Fig. 11, the discrepancy of the strain energy under bi-axial BCs between the volume-average method and the energy method is very small. Here, three 4-element models are established.

On one hand, the 4-element models show a possible big deviation of the strain energy between the two methods, which represents the need of performing numerical bi-axial tests. On the other hand, different models may have different inhomogeneity of the strain distribution, which we think results in the discrepancy between the two methods.

The models are shown in Fig. 13. Four C3D8I elements are utilized to form the models. Each element is assigned specific material properties that are preset in Table 2. The material properties are set to present the supposed discrepancy between $d_1$ and $d_2$ according to their expressions analyzed in Eqs. (31) and (32).

The results of these 4-element models are shown in Table 3. From the results, it can be found that the deviation between $d_1$ and $d_2$ can be larger than 50% in Model 3. However, since $C_{12}$ is about one sixth of $C_{11}$ and $C_{12}$, the deviation of the strain energy reduces to 7.28%. The 4-element models can prove the discrepancy of the strain energy under bi-axial BCs.

![Fig. 13 Four-element model composed of 4 materials.](image)
It can also be seen that the deviation between $d_1$ and $d_2$ varies in the three models as well as the deviation of the strain energy. The maximum and minimum strains are also listed in Table 3. It can be seen that in the 4-element models, the deviation between $d_1$ and $d_2$ increases with the inhomogeneity of strain increasing.

From these results, we get:

1. The discrepancy between the two methods varies with different problems.
2. The inhomogeneity of the strain distribution influences that discrepancy.

5. Conclusions

1. An equality for the volume-average method and the corresponding homogeneous solid. Analysis has been done on the discrepancy between the energy method and the volume-average method. Numerical results show that the energy method obtains values of effective properties closer to experimental results for in-plane moduli.


