Nonlinear mode interactions in the wake of a medium height roughness element

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Abstract

The present study is devoted to nonlinear (resonant) mode interactions in the wake of a medium height, cylindrical roughness element, which is placed in a laminar, airfoil boundary layer. The roughness element causes a considerable mean flow distortion in the near wake centerline region and two counter-rotating vortex pairs arise in the outer spanwise domain. In the streamwise evolution the mean flow stabilizes and a spanwise uniform flow is recovered in the far wake. Upstream of the roughness controlled, low-amplitude Tollmien-Schlichting (TS) wave modes are excited in the upper branch unstable region. The interference of the excited, 2D fundamental modes with the roughness results in the excitation of oblique modes in a broad spanwise wave number range. Nonlinear interactions in-between the (oblique) fundamental modes in the destabilized near wake lead to the generation of primary, low-frequency modes at the difference frequencies of the fundamental modes, before the fundamental modes can recover linear stability characteristics in the far wake. In contrast, the primary, subharmonic-type modes experience a nonlinear growth, which is a result of frequency and spanwise wave number detuned resonant mode interactions with the 2D fundamental modes. The resonant growth of the primary interaction modes initiates a symmetrization of the spectrum in the low-frequency, subharmonic range. That is, symmetric subharmonic-type (secondary) modes are resonantly amplified in the far wake. Therefore, although the fundamental modes recover linear stability characteristics with the stabilization of the mean flow, resonant mode interactions in the low-frequency, subharmonic range initiate the onset of boundary layer transition in the far wake of the medium height roughness element.

Keywords: Laminar boundary layer; controlled disturbance excitation; medium roughness height element; resonant mode interactions;

1. Introduction

Roughness induced laminar to turbulent transition sustains a topic of high interest as a detailed understanding is crucial for the design of efficient airfoils and in recent times, especially, for the development of wind turbine blades. In general, roughness elements can be distinguished depending on their shape into two-dimensional elements, isolated three-dimensional elements\textsuperscript{1} and distributed (sandpaper) roughness\textsuperscript{2}. Additionally, the flow phenomena and, therewith, the transition mechanism downstream of the roughness strongly depend on the particular roughness height. For low roughness heights (i), the mean flow distortion in the roughness wake is small and external disturbances - such as free stream turbulence or sound - couple into the boundary layer mainly via a linear receptivity process at the roughness\textsuperscript{3}. In the medium height range the receptivity process at the roughness becomes nonlinear and a substantial mean...
flow distortion downstream of the roughness element alters the stability characteristics of the boundary layer flow. For a single, cylindrical roughness element (as used in the present study), a horseshoe shaped vortex wraps around the front side of the element and a pair of spiral filaments rises vertically in the wake leading to a counter-rotating vortex pair on each side of roughness centerline\(^1\).\(^4\). However, in the far wake these vortical structures eventually decay so that a spanwise uniform mean flow is recovered. The nonlinear disturbance growth, that is initiated by the boundary layer roughness interaction, is most distinctive in the near wake, since the stabilization of the mean flow in the streamwise evolution counteracts the unsteady disturbance growth\(^5\). In the far wake, the disturbances excited by the boundary layer roughness interference decrease without causing transition leading to a critical roughness Reynolds number \(Re_{h,crit}\) for isolated three-dimensional roughness elements, below which no significant effect on transition is expected\(^1\).\(^6\). In contrast, if the critical roughness Reynolds number is exceeded, that is for high roughness elements (iii), a highly nonlinear disturbance evolution and pronounced vortical structures in the immediate wake of the element are associated with bypass transition in the near roughness vicinity\(^7\).

The present study is devoted to the evolution of controlled Tollmien-Schlichting (TS) wave modes and the subsequent mode interactions in the wake of a single roughness element in the medium height range. Previous investigations in this field addressed the local receptivity of a TS-wave scattering at a three-dimensional roughness\(^8\). For a low-amplitude TS-wave mode at the roughness subharmonic modes prevail within secondary instability, whereas the fundamental mode is dominant for large TS-wave amplitudes. For roughness elements in the medium height range, the scattering of an initially two-dimensional TS-wave at a cylindrical roughness element leads to the formation of pronounced oblique modes at the fundamental mode frequency\(^9\). Experimental investigations have shown that these spanwise structures decay in the streamwise evolution for low-amplitude TS-wave modes, in contrast to a continuous growth of the oblique modes in the roughness wake for large initial TS-wave amplitudes at the roughness position\(^10\). Dominant spanwise modes were observed for wave propagation angles close to \(\theta = 45^\circ\) and the interaction with the roughness element was found to be most pronounced for instability modes near the upper branch of the neutral stability curve\(^11\).

The present paper describes the evolution of three simultaneously excited, low-amplitude TS-wave modes in the wake of the cylindrical, medium height roughness element. The focus is set on the nonlinear mode interaction in-between the fundamental modes and the subsequent resonant amplification of low-frequency, subharmonic-type modes, which are a consequence of the nonlinear excitation of the oblique fundamental modes at the roughness. The paper is organized as follows: Sec. 2 is devoted to the experimental setup and the methods of data acquisition, results are presented in Sec. 3 and, finally, Sec. 4 contains the concluding remarks.

2. Experimental Setup and Data Acquisition

The experiments were performed in the Laminar Wind Tunnel of the IAG\(^12\), which is of Eiffel type. An effective contraction ratio of 20:1 and several screens in the inlet section lead to a low longitudinal turbulence level of \(Tu_c = 0.02\%\) for a frequency range of \(f = 10 – 5000\) Hz and a free stream velocity of \(u_{\infty} = 30\) m/s. The closed test section has a cross section of \(0.73 \times 2.73\) m\(^2\) and a length of 3.15 m. The diffuser section is equipped with noise absorbing foam substantially reducing the background noise level in the test section\(^13\). During the measurements the free stream velocity and not the Reynolds number was kept constant due to the quadratic influence of the non-dimensional frequency parameter \(F = 2\pi f\nu/u_{\infty}^2\) on the free stream velocity (with \(\nu\) being the kinematic viscosity).

In order to achieve a close alignment of the experimental with operational conditions of wind turbines the experiments are performed on a specially designed airfoil section (BE72)\(^14\). For angles of attack corresponding to the upper edge of the low drag bucket the pressure gradient in the leading edge region of the BE72 airfoil section is similar compared with the NACA 643 – 418 airfoil, which can be regarded as a typical airfoil for wind turbine applications. However, the pressure distribution in the trailing edge section of the BE72 was modified to reduce the model chord to \(c = 2.4\) m. Therewith, experiments can be conducted at a typical Reynolds number for wind turbine applications \((Re = 6 \cdot 10^6)\) and at low free stream velocity \((u_{\infty} = 20\) m/s\). For the present study, the pressure gradient is close to zero in the entire measurement domain resulting in an almost constant boundary layer edge velocity of \(u_0 = 28.3\) m/s. Subsequently, the boundary layer shape factor is nearly constant at \(H_{12} = 2.59\). The boundary layer displacement thickness and the Reynolds number based on displacement thickness at the roughness position are \(\delta_{1,ref} = 0.72\) mm and \(Re_{\delta_{1,ref}} = \delta_{1,ref} u_0 / \nu = 1350\), respectively. Below, the wall-normal coordinate \(y\) and the roughness height \(h\) are
normalized with the displacement thickness at the roughness position \( \delta_{1,\text{ref}} \) and the streamwise \( s \) and spanwise \( z \) coordinate are normalized with the roughness diameter \( d_r \) (see next paragraph).

A cylindrical roughness element with a diameter \( d_r = 20 \text{ mm} \) is integrated surface flush into the airfoil model at a streamwise location of \( x/c = 0.15 \). The roughness height can be adjusted in a range \( 0 < h/\delta_{1,\text{ref}} < 2 \) with a linear actuator and the height is monitored based on laser-triangulation with a linearity of ±3 \( \mu \text{m} \) corresponding to 0.4 percent of the boundary layer displacement thickness at the roughness position. Upstream of the roughness controlled TS-wave modes are excited at \( x/c = 0.1 \) by a small slit (width 0.2 mm), which is integrated into the model surface. Beneath the slit, 128 tubes are connected to individually controllable loudspeakers, which are driven by a custom signal generator\(^{15} \). In order to ensure phase synchronism for the disturbance excitation the same clock is used for all 128 channels of the signal generator. In the reference case (zero roughness height), the streamwise evolution of the excited TS-waves complies well with linear theory within the measurement domain\(^{16} \). At the roughness position, the standard deviation of the TS-wave amplitude and phase in spanwise direction is 8% of the mean amplitude and 6.8° for the TS-wave phase, i.d. 1.9% of 360°.

Phase-locked (with respect to the disturbance excitation) hot-wire measurements were performed with a single wire probe of type 55P15. The probe has a 5 \( \mu \text{m} \) gold-plated tungsten wire and was operated with a DISA 55M10 bridge with a constant overheat ratio of 1.8. Prior to AD conversion the hot-wire signal was split into AC and DC part. For the DC part an active low-pass filter with a cut-off frequency of \( f_{\text{cl}} = 1 \text{ Hz} \) was used. The AC part was high-pass filtered with a passive RC-filter (\( f_{\text{ch}} = 50 \text{ Hz} \)). The AD-converter is equipped with an internal \( f_s = 40 \text{ kHz} \) low-pass filter, which minimizes the effect of very high-frequency noise. No additional anti-aliasing filter was used due to a high sampling rate (\( f_s = 15625 \text{ Hz} \)) and a strong decay of the spectral density in the high-frequency range. AC and DC part of the signal are acquired at the same time with a 18-Bit AD converter (NI USB-6289).

For each measurement point a total of 32768 samples was recorded. Prior to a Fast Fourier transformation (FFT) the fluctuation velocities are split into 8 blocks of 4096 samples and ensemble averaged in the temporal domain to increase the signal-to-noise ratio. For the FFT a rectangular window without overlap was applied, since the excited modes exactly correspond to single Fourier series coefficients. Spanwise boundary layer measurements were interpolated to an equidistant grid with a constant spanwise domain of \(-7.5 < z/d_r < 7.5 \) to obtain spanwise wave number spectra via complex FFT. Since the disturbance modes excited by the roughness spread in a wedge shaped region in the downstream evolution (compare e.g. Fig. 3), the amplitudes of the oblique modes have been corrected for the influence of the (in streamwise direction) increasing spanwise extend, which is affected by the boundary layer roughness interaction. The spanwise wave number \( \beta = 2\pi/\lambda_z \) is normalized with the roughness diameter \( d_r \) (with \( \lambda_z \) being the spanwise wavelength). That is, \( \beta d_r = 2\pi \) corresponds to a wavelength equal to the roughness diameter. Further details on the experimental setup and the methods of data processing are found in Plogmann et al.\(^{16} \).

### 3. Results and Discussion

#### 3.1. Mean flow evolution

Fig. 1a shows the mean flow evolution downstream of the medium height roughness element at \( h/\delta_{1,\text{ref}} = 0.3 \) in a streamwise-spanwise plane at \( y/\delta_{1,\text{ref}} = 0.9 \). The associated roughness Reynolds number is \( Re_h = 70 \) (with \( Re_h = u_h h/\nu \) and \( u_h \) being the velocity at the height of the roughness in reference case). In the near wake, a considerable velocity deficit is observed, which is most distinct in the centerline region. Further downstream the mean flow distortion in the centerline region decreases and distinct high speed streaks arise at spanwise positions of \( z/d_r = \pm 0.5 \) corresponding to the location of the roughness edges (see also Fig. 1b). Here, \( u_w \) is the mean flow velocity at the undisturbed spanwise edges of the measurement domain. At the outer spanwise edge of each high speed

Fig. 1. Mean flow evolution in the streamwise-spanwise plane at \( y/\delta_{1,\text{ref}} = 0.9 \) (a) and streamwise development of streak amplitude (b) for a roughness height of \( h/\delta_{1,\text{ref}} = 0.3 \).
Fig. 2. Boundary layer spectra at $y/\delta_{1,ref} = 0.9$ with reference spectrum $(h/\delta_{1,ref} = 0)$ indicated by dotted line at $s/d_r = 3.75$, shifted by one decade per $\Delta s/d_r = 1$ (a), amplitude eigenfunctions of mode $f_{f1} = 549$ Hz with reference eigenfunction for $h/\delta_{1,ref} = 0$ at $s/d_r = 10.75$ indicated with symbols and mean velocity profile at $s/d_r = 3.75$ (b) and streamwise evolution of maximum wall normal amplitude $A_{m}$ for fundamental modes $f_{f1-3}$ and of shape factor (c) on roughness centerline (frequencies of fundamental $(f_{f})$, harmonic $(f_{h})$, sum $(f_{s})$, difference $(f_{d})$ and secondary interaction $(f_{f,\Delta})$ modes indicated in (a) by thin dashed, thick solid, thick dashed, dotted and thin solid lines, respectively).

3.2. Fundamental mode evolution

Upstream of the roughness a trio of two-dimensional TS-wave modes has been excited in the upper branch unstable region, that is at frequencies $f_{f1} = 549$ Hz, $f_{f2} = 606$ Hz and $f_{f3} = 686$ Hz (see Fig. 2a). The streamwise wavelength of the three fundamental modes is in the same range as the roughness diameter, that is $\alpha \delta / d_r = 2.2\pi$ for $f_{f3} = 686$ Hz and $\alpha \delta / d_r = 1.9\pi$ for $f_{f1} = 549$ Hz. A stability diagram can be found in Plogmann et al. The maximum (wall-normal) RMS amplitude of each fundamental mode is in the same range as the roughness diameter, that is

$$A_{m}/u_{\infty} = 0.02\%$$

resulting in a total RMS amplitude of $A_{m}/u_{\infty} = 0.035\%$ at the roughness position. Fig. 2c shows the fundamental mode growth for $h/\delta_{1,ref} = 0.3$ in comparison to the zero-roughness height case. Here, the growth is based on the maximum wall-normal amplitude on the roughness centerline (compare Fig. 2b). In the reference case ($h/\delta_{1,ref} = 0$) the growth of the 2D fundamental modes matches with linear theory up to the end of the measurement domain. In contrast, the boundary layer roughness interaction initiates a nonlinear growth of the fundamental modes in the near wake. This growth is associated with the formation of a second near wall peak close to $y/\delta_{1,ref} = 0.9$ in the amplitude eigenfunction (Fig. 2b). This amplitude peak is located near the critical layer as can be seen from the comparison to the mean flow velocity profile at $s/d_r = 3.75$ and can be linked to the formation of oblique (fundamental) modes in the near wake. From the spanwise amplitude distribution near the critical layer ($y/\delta_{1,ref} = 0.9$), which is exemplary shown for the mode $f_{f1} = 549$ Hz in Fig. 3a, it becomes obvious that the growth is most distinct in the destabilized, low speed centerline region. In contrast, amplitude minima are present at $z/d_r = \pm 0.5$, that is at the position of the high speed mean flow streaks showing their stabilizing effect on the disturbance evolution in the near wake (compare Fig. 1a). Further outwards (in spanwise direction) the evolution of the fundamental modes is not considerably affected in the near wake ($s/d_r < 4$). The spanwise modulated fundamental modes in the near wake are associated with a broad spectrum of oblique modes in agreement with previous findings (Fig. 3b). The peak in the spanwise wave number spectrum is located near $\beta d_r = 5 - 6$ corresponding to a spanwise wavelength, which is close to the roughness diameter. Note, the spanwise wave number spectra are averaged with respect to $\pm \beta$ if not stated otherwise, since the disturbance evolution in the roughness wake is symmetric with regard to the roughness centerline.

The nonlinear growth of the fundamental modes in the near wake levels off with the stabilization of the mean flow as can be seen from the comparison to the evolution of the shape factor on the roughness centerline (Fig. 2c, top).
Hence, a local amplitude maximum is present at $s/d_r \approx 4$, which appears rather independent of the fundamental mode frequency for the excited modes $f_{j1-3}$. Further downstream the fundamental modes are slightly damped, before they grow/decay with a very similar characteristic as in the undisturbed reference case at $h/\delta_{1,ref} = 0$. However, the fundamental mode amplitude is increased by $\Delta n = 0.6-1$ compared with the reference case (with $\Delta n = \ln(A(h/\delta_{1,ref} = 0.3)/A(h/\delta_{1,ref} = 0))$ at $s/d_r = 10.75$). The amplitude evolution in the streamwise-spanwise plane in Fig. 3a further reveals that the minima at $z/d_r \approx \pm 0.5$ start to spread outwards in a wedge-shaped region when the amplitude of the high speed mean flow streaks decreases in the far wake (compare Fig. 1b). In-between these two amplitude minima a wave-type structure is observed. The evolution of the associated spanwise wave number spectra reveals that the strongest growth is present for oblique modes close to $\beta d_r = 5 - 6$. That is, the most dominant spanwise modes are in the range of the streamwise fundamental mode wavelength leading to wave propagation angles near $\Theta = 45^\circ$ in compliance with previous observations. However, in the streamwise evolution the oblique modes are damped with the relaxation of the mean flow towards a spanwise uniform flow. Hence, no relevant (compared with the 2D mode) oblique modes from the TS-wave roughness interaction persist into the far wake. In compliance, the second near wall peak in the TS-wave eigenfunction on the roughness centerline decreases and the amplitude function approaches the linear eigenfunction of the undisturbed reference case (Fig. 2b).

3.3. Nonlinear mode interactions

The nonlinear growth of the fundamental modes in the destabilized near wake leads to mode interactions as can be seen in the boundary layer spectrum at $s/d_r = 3.75$ and $s/d_r = 10.75$ (Fig. 2a). The mode interactions become evident in higher harmonic modes ($f_{h}$) of the excited fundamental modes ($f_{j}$) at frequencies $f_{h} = n \times f_{j}$ (with $n = 2, 3, ..$). The higher harmonic mode frequencies are indicated by thick solid lines on top of Fig. 2a. Fig. 4a shows the streamwise evolution of the second ($n = 2$) harmonic mode amplitude in relation to the fundamental mode amplitude. Here, the RMS amplitude of all modes belonging to a certain group is depicted. That is, the fundamental mode amplitude is e.g. $\Sigma f_{j} = \sqrt{f_{j1}^2 + f_{j2}^2 + f_{j3}^2}$. The harmonic mode amplitude develops with the same characteristic as the fundamental mode amplitude, that is the harmonic modes grow/decay with the fundamental ones. However, compared with the fundamental mode the harmonic mode amplitude remains substantially (2 orders of magnitude) lower within the entire measurement domain showing that the higher harmonic modes play an underpart in the first nonlinear stages of transition downstream of the medium height roughness element.

In the near wake, additional interaction modes arise at sum ($f_{2}$) and difference ($f_{3}$) frequencies of the three fundamental modes. The sum and difference modes are indicated by dashed and dotted lines in Fig. 2a, respectively. The sum modes exhibit a similar growth behavior as the fundamental and its higher harmonic modes (Fig. 4a). Previous investigations have shown that the growth of the sum modes can be explained by a quadratic wave interaction of the respective fundamental modes. This quadratic interaction seems to be slightly more effective since the amplitude of the sum modes is increased compared with the second harmonic mode amplitude. However, the amplitude of the sum modes is still considerably lower with regard to the fundamental mode amplitude. Moreover, the decay in the far wake lead to similar amplitude levels for the sum and the harmonic modes, which are only marginally increased compared with the reference case at $h/\delta_{1,ref} = 0$. Hence, the sum modes are (as the harmonic modes) of minor importance in the transition process downstream of the medium height roughness (see also Fig. 2a). In contrast, for the low-frequency difference ($f_{3}$) modes a continuous growth is present in the near wake, which levels off only significantly further.
downstream compared with the evolution of the sum and the harmonic modes. Accordingly, the amplitude of the difference modes reaches the same order of magnitude as the fundamental modes’ amplitude in the far wake (compare also Fig. 2a). Note, the constant amplitude level of the difference modes for \( s/d_e < 3 \) in Fig. 4a is due to the increased noise level in the low-frequency range, which is caused by wind tunnel background noise and vibrations of the hot-wire traverse (see Fig. 2a). Therefore, low-frequency modes can only be tracked at higher amplitudes compared with the high-frequency modes.

Below, further light is shed on the nature of the low-frequency difference modes. Fig. 4b shows the wall-normal amplitude and phase function exemplarily for the difference mode \( f_{A31} = f_3 - f_1 = 137 \) Hz on the roughness centerline in the far wake. In the amplitude function a pronounced peak is observed near the critical layer \( (y/\delta_{1,ref} = 0.9 - 1) \), similar as observed in the fundamental mode eigenfunction in the near wake (compare Fig. 2b). A good match is obvious for the wall-normal amplitude and phase function of the difference mode with regard to the linear eigenfunction of an oblique mode with \( \beta = \alpha_r \) revealing a TS-wave type character of the difference modes. The oblique nature of the difference modes is also obvious in the spanwise amplitude and phase distributions near the critical layer (see Plogmann et al.16). The difference modes start to grow in the centerline region and spread outwards in a wedge shaped region, which is limited in spanwise direction by the outer amplitude minima in the fundamental mode evolution (compare Fig. 3a). Fig. 4 shows the associated spanwise wave number spectra for the difference mode \( f_{A31} = 137 \) Hz. In the near wake difference modes arise in a broad spanwise wave number range with the peak located near \( \beta_{d_e} = 9 \). Hence, in comparison to the fundamental mode spectra in the near wake (Fig. 3b) the peak is shifted towards higher spanwise wave number modes for the difference modes at first. However, in the streamwise evolution the growth of the high spanwise wave number modes levels off and in the far wake a damping is observed. In contrast, lower spanwise wave number modes are stronger amplified and, accordingly, the maximum is shifted to lower spanwise wave numbers in the downstream evolution. Far downstream, the maximum is observed close to \( \beta_{d_e} = 5 - 6 \) and a second less distinct peak becomes obvious close to \( \beta_{d_e} = 12 \). Hence, in the far wake dominant oblique modes are present in a similar spanwise wave number range for the fundamental and the associated low-frequency, difference modes (compare also Fig. 5c).

In the streamwise evolution further distinct spectral peaks at frequencies \( f_{f,\Delta} \) in the boundary layer spectrum show the development of secondary interaction modes (see Fig. 2a). Here, secondary interaction modes are referred to as modes, which imply the interference of a low-frequency difference (primary interaction) mode with a fundamental mode, that is \( f_{f,\Delta} = f_f \pm f_\Delta \). Thin solid lines on the top of Fig. 2a indicate all secondary interaction modes that can theoretically arise. The comparison with the boundary layer spectrum reveals that secondary interaction modes are most pronounced in the low-frequency range \( (f = 400 - 500 \) Hz), that is for modes, which are based on an interaction with the fundamental mode \( f_{f1} \). In contrast, interactions of the primary with the higher frequency fundamental modes cannot be observed likely due to the considerable decay of the fundamental modes \( f_{f2} \) and \( f_{f3} \) with the relaxation of the mean flow in the far wake (compare Fig. 2c).

Fig. 4a shows the RMS amplitude of all secondary interaction modes in comparison with the fundamental and the primary interaction mode amplitudes. For the secondary modes, a continuous growth is observed in the downstream evolution. However, in the near wake discrete peaks at the secondary mode frequencies are not present in the boundary layer spectrum (Fig. 2a). Instead the growth in the \( f_{f,\Delta} \) frequency bands in the near wake can be linked to the naturally
arising disturbances in the near wake (compare Fig. 2a and Plogmann et al. [16]). In contrast, further downstream \((s/d_0 > 6)\) the growth in the \(f_{fA}\) frequency bands can mainly be attributed to the discrete growth of the secondary interaction modes. In the far wake, the amplitude of the secondary interaction modes is still an order of magnitude lower compared with the fundamental mode amplitude. However, the secondary interaction modes lead to a progressively filling up disturbance spectrum in the far wake, especially in the low-frequency, unstable TS-wave range (Fig. 2a).

The secondary interaction modes start to grow in the centerline region, in which the highest amplitudes are present for the fundamental and the difference modes, respectively. In the streamwise evolution the secondary interaction modes spread outwards in a wedge-shaped region, which is limited by the amplitude minima in the fundamental mode evolution [16]. Fig. 5a shows the wall-normal amplitude and phase function exemplarily for the secondary interaction mode \(f_{f1,31} = f_{f1} - f_{31} = 412\) Hz on the roughness centerline. The wall-normal amplitude maximum is observed near \(y/\delta_{1,ref} = 0.9\). That is, the secondary interactions are most distinct near the critical layer in agreement with the observations for the primary modes (compare Fig. 4b). Moreover, good agreement with regard to the linear eigenfunction of a mode with \(\beta = \alpha_r\) reveals a TS-wave type character of the secondary interaction modes. The \(f_{fA}\) modes become most distinct in a spanwise wave number range of \(\beta \delta_r = 4 - 7\) in the downstream evolution, whereas a significant growth at higher spanwise wave numbers \((\beta \delta_r > 10)\) - as seen for the primary modes in the near wake - cannot be observed (see Fig. 5b). Hence, the highest amplitudes are observed in the same spanwise wave number range as for the fundamental and the primary interaction modes in the far wake (Fig. 5c).

### 3.4. Resonant mode interactions in the low-frequency, subharmonic range

The discussion above has shown that the interference of the initially two-dimensional TS-wave modes with the cylindrical roughness element results in the formation of oblique modes at the fundamental mode frequencies. Fig. 6 shows the evolution of the amplitude (a) and the phase speeds (b) for the fundamental and the interaction modes exemplary for the modes \(f_{f1}, f_{f3}, f_{31}\) and \(f_{f1,31}\) at a spanwise wave number of \(\beta \delta_r = 5.5\), which is close to the maximum amplitude of the oblique fundamental and interaction modes, respectively (compare Fig. 5c). Here, the amplitude and phase evolutions are based on spanwise wave number spectra near the critical layer \((y/\delta_{1,ref} = 0.9)\) corresponding to the peak in the eigenfunction of the oblique modes with \(\beta \delta_r = 5.5\) (compare Fig. 2b, Fig. 4b and Fig. 5a). The decreased amplitude of the 2D mode with regard to its maximum, which is located closer to the wall (see Fig. 2b), is corrected for by considering local linear eigenfunctions.

For the 2D fundamental modes a weak nonlinear growth is observed in the near wake, but for \(s/d_0 > 6\) linear growth rates are recovered. The phase speed of the 2D fundamental modes is slightly increased in the near wake, but matches with linear theory in the far wake. The oblique fundamental mode \(f_{f1}\) with \(\beta \delta_r = 5.5\) experiences a strong nonlinear growth in the near wake, but similar as observed for the 2D fundamental modes linear growth rates and phase speeds are recovered in the far wake. That is, due to the low initial amplitudes at the roughness, the fundamental modes can recover linear stability characteristics with the relaxation of the boundary layer towards the undisturbed base flow as already hinted at in Fig. 2c. Moreover, the phase speeds of the fundamental mode \(f_{f3}\) and the oblique mode \(f_{f1}\) are very close in the far wake, but the linear growth gives evidence that no resonant interactions occur. However, the nonlinear interactions in-between the oblique fundamental modes in the near wake lead to the generation of the low-frequency difference modes (compare Sec. 3.3). After their generation, those primary modes...
can, in principle, also be understood as subharmonic-type modes with frequency and spanwise wave number detuning (as argued below). Hence, the nonlinear interactions in the near roughness wake lead to favorable initial conditions for a subharmonic-type breakdown scenario in the later stages of transition downstream of the medium roughness height element.

In this context, Fig. 7a shows the dispersion curves exemplary for the fundamental mode $f_{s1}$ and the corresponding subharmonic mode $f_{s31} = f_{s1}/2$ as the fundamental mode $f_{s1}$ has the highest amplitude and growth rates in the streamwise evolution downstream of the roughness. In order to satisfy the condition of phase synchronism for a tuned, resonant Craik triad the spanwise wave number of the subharmonic mode $f_{s31}$ would be $\beta_{s31} = \beta_{s1}/2$. However, it is known that resonant interactions can still be present for large frequency (up to 90%) and spanwise wave number detuned subharmonic modes (e.g. Borodulin et al.19). In the present case, the frequency detuning would be $\Delta f_{s31}/f_{s31} = -50\%$ for the oblique $f_{s31}$ with regard to the subharmonic mode $f_{s1}$ (with $\Delta f_{s1} = f_{s31} - f_{s1}$). In the spanwise wave number spectrum of mode $f_{s31}$ maximum amplitudes are observed for $\beta_{s1} = 4 - 7$ being in the same range as the tuned spanwise wave number $\beta_{s1}$ of the subharmonic mode $f_{s31}$ (compare Fig. 7a and b). For the maximum near $\beta_{s1} = 5.8$ the spanwise wave number detuning would be $\Delta \beta = (\beta_{s1} - \beta_{s31})/\beta_{s1} = 40\%$.

For the low-frequency, subharmonic-type mode $f_{s31}$ a nonlinear growth is observed for $s/d_r > 4$. The phase speed of the $f_{s31}$ mode deviates considerably from the linear phase speed, but matches well with the phase speed of the 2D fundamental mode $f_{s1}$ up to the end of the measurement domain. That is, the low-frequency difference mode synchronizes with the 2D fundamental mode indicating that the nonlinear growth of the low-frequency difference modes is caused by a resonant mode interaction in contrast to the non-resonant mode interactions in-between the (oblique) fundamental modes in the near wake. The amplitude of the pure 2D fundamental mode is, however, still below the previously observed threshold of $A/\mu_8 \approx 0.002$ for subharmonic resonance to set in (e.g. Kachanov and Levchenko20). This suggests that the onset of the resonant mode interactions is caused by (local) amplitude peaks, which result from the superposition of the 2D and quasi 2D (low spanwise wave number) fundamental modes (compare Fig. 3a and b).

In the far wake ($s/d_r > 7$) the growth of the $f_{s31}$ mode levels off and shows a similar growth behavior as predicted by linear theory ($s/d_r > 9$). In compliance, it is seen, that the 2D and quasi 2D fundamental modes recover linear stability characteristics, so that their superimposed amplitude growths considerably less (compared with the near wake) or even decreases slightly as observed in the centerline region (Fig. 3a and also Plogmann et al.21). That is, the resonant mode interactions become weaker in the far wake with the less distinct growth/decay of the driving 2D and quasi 2D fundamental modes.

Previous studies have shown that if a single frequency detuned subharmonic mode is excited (e.g. $f_{exc} = f_s + \Delta f$) an additional, symmetric (with respect to the exact subharmonic) mode appears at a frequency $f_{sym} = f_s - \Delta f$ and tends to reach a similar amplitude level as the excited mode. The initially excited subharmonic mode can be understood as an oscillation at the exact subharmonic frequency, but with its phase depending on time. When resonant interaction exists, the drifting phase leads to a temporal modulation of the excited mode resulting in the symmetrization of the frequency spectrum in the subharmonic range. In the present case, the initial subharmonic-type modes are the primary interaction modes, e.g. $f_{s31} = f_{s1} + \Delta f_{s1}$ (see discussion above). The corresponding symmetric modes would, then, be the secondary interaction modes, e.g. $f_{s31} = f_{s1} - \Delta f_{s1}$, which become obvious in the far wake (compare Sec. 3.3).

The similar spectral content of the spanwise wave number spectra of the $f_{s31}$ and the $f_{s1A31}$ mode (Fig. 5c) gives a first indication that the secondary modes can be viewed as symmetric subharmonic-type modes. In Fig. 6a it is seen that the symmetric $f_{s1A31}$ mode starts to arise from the background noise level for $s/d_r > 4$. The comparison to
linear theory reveals that the growth is considerably more pronounced in the entire downstream evolution. Moreover, the \( f_{1,31} \) mode phase speed deviates from linear theory, but instead is very similar to the phase speed of the 2D fundamental mode \( f_1 \) and the initial subharmonic-type mode \( f_{31} \) (Fig. 6c). This gives further evidence, that the nonlinear growth of the secondary interaction mode \( f_{1,31} \) in the far wake is caused by a symmetric (with regard to the \( f_1 \) mode) resonant mode interaction, which, in turn, can be linked to the phase-locked interaction of the fundamental mode \( f_1 \) and the primary interaction mode \( f_{31} \). Based on Fig. 6a it is further obvious that the growth of the symmetric mode \( f_{1,31} \) is considerably more distinct in the far wake, whereas the growth of the initial subharmonic mode \( f_{31} \) levels off with the 2D fundamental mode \( f_1 \). This might be related to the lower amplitude of the symmetric mode \( f_{1,31} \), which has not yet reached the level of the initial subharmonic-type mode \( f_{31} \).

In Fig. 2a it is obvious that all symmetric modes with regard to the exact subharmonic frequency \( f_1 \) are present, that is \( f_{1,31} = 412 \text{ Hz} \), \( f_{1,32} = 469 \text{ Hz} \) and \( f_{1,321} = 492 \text{ Hz} \). In contrast, a significant growth at the higher secondary interaction mode frequencies, which would be symmetric modes with regard to the subharmonic frequencies \( f_2 \) and \( f_3 \), cannot be observed (with \( f_{s2} = f_{2,2} / 2 \) and \( f_{s3} = f_{2,3} / 2 \)). In compliance, it was seen that the \( f_{31} \) mode does not synchronize with the 2D fundamental mode \( f_{1,2} \) or \( f_{1,3} \), but instead with the lower frequency fundamental mode \( f_1 \), which exhibits the highest amplitude and growth (of the 2D fundamental modes) in the far wake (e.g. Fig. 6a). These observations agree with previous findings, which revealed that in the presence of multiple 2D fundamental waves resonant interactions are driven by the lower frequency fundamental modes in the streamwise evolution as they normally exhibit a higher linear growth and, therewith, higher amplitudes\textsuperscript{22}.

Finally, Fig. 7b shows the streamwise evolution of the subharmonic mode in comparison to the fundamental mode amplitude on the roughness centerline. Here, e.g. the subharmonic \( f_{31} \) mode amplitude is the sum of the initial \( f_{31} \) and the symmetric \( f_{1,31} \) mode amplitude. For \( s/d_r > 3 \) (that is when the amplitude of all subharmonic-type modes is well above the electronic noise level), the three subharmonic modes develop very similar and only minor differences are obvious in the streamwise amplitude evolution. This implies that the frequency detuning, which is \( \Delta f_{32,1} / f_1 = \pm 71\% \) and \( \Delta f_{32,1} / f_1 = \pm 79\% \) for the \( f_{32} \) and \( f_{2,1} \) mode, respectively, has no significant influence on the amplification of the subharmonic-type modes in the investigated range of frequencies. Moreover, a similar evolution of the subharmonic modes further supports the argument that the subharmonic growth can mainly be associated with the same 2D fundamental mode \( f_1 \) at least in the far wake, whereas the fundamental modes \( f_{1,2} \) and \( f_{1,3} \) play an underpart for the resonant amplification of the subharmonic-type modes, since their amplitudes decay considerably in the far wake.

4. Conclusion

The flow downstream of a medium height roughness element placed in a laminar airfoil boundary layer has been experimentally investigated with hot-wire anemometry. In the near wake centerline region a considerable mean flow distortion is present and two counter-rotating vortex pairs are observed at the edges of the roughness, but in the far wake the mean flow is stabilized and a spanwise uniform flow is recovered. Upstream of the roughness two-dimensional (2D), fundamental TS-wave modes of low amplitude have been excited in the upper branch unstable region according to linear theory. The interference of the 2D fundamental modes with the roughness results in a nonlinear initiation of a broad spectrum of oblique modes at the fundamental frequencies. The subsequent nonlinear interactions in-between the (oblique) fundamental modes in the destabilized near wake lead to the formation of low-frequency, subharmonic-type modes at the difference frequencies of the fundamental modes. In the far wake the 2D and oblique fundamental modes can recover linear stability characteristics with the stabilization of the mean flow. In contrast, the low-frequency difference (primary interaction) modes, which have a modal, TS-wave type character, experience a nonlinear growth. A synchronization of these low-frequency, subharmonic-type modes with
the 2D fundamental modes gives evidence that the nonlinear growth is caused by frequency and spanwise wave number detuned resonant mode interactions. In the streamwise evolution these resonant mode interactions result in a symmetrization of the frequency spectrum in the low-frequency range with regard to the exact subharmonic frequency. That is, symmetric (secondary), subharmonic-type modes arise and progressively fill-up the disturbance spectrum in the low-frequency range, although the fundamental modes recover linear stability characteristics in the far wake. Hence, the nonlinear excitation of the oblique fundamental modes at the roughness and their subsequent nonlinear interactions lead to the generation of subharmonic-type modes, which are then resonantly amplified by an interaction with the 2D fundamental modes and, thereby, dominate the first nonlinear stages of transition in the far wake of the medium height roughness element.

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**References**

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