A comparative study of numerical methods for solving the generalized Ito system

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Received 13 February 2013; revised 28 May 2013; accepted 8 June 2013
Available online 30 July 2013

KEYWORDS
The variation iteration method; The Laplace decomposition method; The Pade approximation; Homotopy perturbation method; The generalized Ito system

Abstract
This paper is devoted to the numerical comparison of methods applied to solve the generalized Ito system. Four numerical methods are compared, namely, the Laplace decomposition method (LDM), the variation iteration method (VIM), the homotopy perturbation method (HPM) and the Laplace decomposition method with the Pade approximant (LD–PA) with the exact solution.

2000 MATHEMATICS SUBJECT CLASSIFICATION:
33F05; 35A15; 35C10; 65M12; 70G75
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1. Introduction

In recent years 1999, the variational iteration method (VIM) was proposed by He in [1–6]. This method is now widely used by many researchers to study linear and nonlinear partial differential equations. The method introduces a reliable and efficient process for a wide variety of scientific and engineering applications, linear or nonlinear, homogeneous or inhomogeneous, equations and systems of equations as well. Many authors [7–11] that this method is more powerful than existing techniques such as the Adomian method [12,16], perturbation method, etc. showed it. The method gives rapidly convergent successive approximations of the exact solution if such a solution exists; otherwise a few approximations can be used for numerical purposes. Another important advantage is that the VIM method is capable of greatly reducing the size of calculation while still maintaining high accuracy of the numerical solution. Moreover, the power of the method gives it a wider applicability in handling a huge number of analytical and numerical applications. Many authors for different cases have obtained some exact and numerical solutions of the generalized Ito system (see [17–20]).

Consider the generalized Ito system [21]:

\[ u_t = v_x, \quad v_t = -2v_{xxx} - 6(uv)_x + aww_x + btp_x + cwp_x + dp_x + fp_x + gp_x, \]
\[ w_t = w_{xxx} + 3uv_x, \quad p_t = p_{xxx} + 3up_x. \] (1)

The general exact solution of above system is:
\[
\begin{align*}
&u(x, t) = \frac{2 - \beta}{3} - 2 \tanh \left( x - t\beta \right)^2, 
&v(x, t) = \frac{\beta^2}{3} - 2 + 4\beta \tanh \left( x - t\beta \right)^2, 
&w(x, t) = \frac{-bf + ag - \beta^2 c_0 + adel}{a(b - c)} 
\end{align*}
\]
\[
\begin{align*}
2 \left( \frac{\beta^2 x - 8af - 9af^2}{\sqrt{b - cc}} + \frac{48f - 9af^2}{\sqrt{b - cc}} \right) \tanh (x - t\beta) 
+ & -be + ad 
\end{align*}
\]
\[
P(x, t) = c_0 - 2 \sqrt{8af + 9af^2} \tanh (x - t\beta),
\]
(2)

For simplicity, we take: \(a = -37, \ b = 2, \ c = \frac{1}{2}, \ d = -1, \ f = 2, \ g = 2, \ c_0 = -1, \ \beta = \frac{1}{2} \). We have:

\[
\begin{align*}
&u(x, t) = \frac{7}{12} - 2 \tanh \left( x - \frac{t}{4} \right)^2, \quad v(x, t) = -\frac{7}{48} + \frac{1}{2} \tanh \left( x - \frac{t}{4} \right)^2, \\
&w(x, t) = -\frac{36}{37} + \frac{1}{6} \tanh \left( x - \frac{t}{4} \right), \quad p(x, t) = 4 + \frac{37}{12} \tanh \left( x - \frac{t}{4} \right),
\end{align*}
\]
(3)

with initial conditions:

\[
\begin{align*}
&u(x, 0) = \frac{7}{12} - 2 \tanh \left( x \right)^2, \quad v(x, 0) = -\frac{7}{48} + \frac{1}{2} \tanh \left( x \right)^2, \\
&w(x, 0) = -\frac{36}{37} + \frac{1}{6} \tanh(x), \quad p(x, 0) = 4 + \frac{37}{12} \tanh(x).
\end{align*}
\]
(4)

The aim of this paper is to use the Laplace decomposition method (LDM), the variation iteration method (VIM), Homotopy perturbation method (HPM) and the Pade approximant (LD-PA) to find the numerical solution of Eq. (1), compare our obtained results with the exact solution, and compute the error.

2. Methods and its applications

2.1. The variational iteration method [3-8]

For the purpose of illustration of the methodology to the proposed method, using variational iteration method [22-26], we write a system in an operator form as:

\[
\begin{align*}
L_u + R_1 (u, v, w, p) + N_1 (u, v, w, p) &= g_1, \\
L_v + R_2 (u, v, w, p) + N_2 (u, v, w, p) &= g_2, \\
L_w + R_3 (u, v, w, p) + N_3 (u, v, w, p) &= g_3, \\
L_p + R_4 (u, v, w, p) + N_4 (u, v, w, p) &= g_4,
\end{align*}
\]
(5)

with initial data

\[
\begin{align*}
u(x, 0) = f_1(x), \quad v(x, 0) = f_2(x), \\
w(x, 0) = f_3(x), \quad p(x, 0) = f_4(x),
\end{align*}
\]
(6)

where \(L_u\) is considered a first-order partial differential operator, \(R_j, 1 \leq j \leq 4, \) and \(N_j, 1 \leq j \leq 4, \) are linear and nonlinear operators respectively, and \(g_1, g_2 \) and \(g_3 \) are source terms. In what follows we give the main steps of He’s variational iteration method in handling scientific and engineering problems. The system (5) can be written as:

\[
u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda_1 \left( L u_n(x) + R_1 (\hat{u}, \hat{v}, \hat{w}, \hat{p}) + N_1 (\hat{u}, \hat{v}, \hat{w}, \hat{p}) \right) \, dt,
\]
\[
v_{n+1}(x, t) = v_n(x, t) + \int_0^t \lambda_2 \left( L v_n(x) + R_2 (\hat{u}, \hat{v}, \hat{w}, \hat{p}) + N_2 (\hat{u}, \hat{v}, \hat{w}, \hat{p}) \right) \, dt,
\]
\[
w_{n+1}(x, t) = w_n(x, t) + \int_0^t \lambda_3 \left( L w_n(x) + R_3 (\hat{u}, \hat{v}, \hat{w}, \hat{p}) + N_3 (\hat{u}, \hat{v}, \hat{w}, \hat{p}) \right) \, dt,
\]
\[
p_{n+1}(x, t) = p_n(x, t) + \int_0^t \lambda_4 \left( L p_n(x) + R_4 (\hat{u}, \hat{v}, \hat{w}, \hat{p}) + N_4 (\hat{u}, \hat{v}, \hat{w}, \hat{p}) \right) \, dt,
\]

where \(\lambda_j, 1 \leq j \leq 4, \) are restricted variations which means \( \lambda_j (0) = 0 . \)

2.2. The Laplace decomposition method (LDM) [27]

In this section, Laplace decomposition method [27-29] is applied to the system of partial differential Eq. (1). The method consists of first applying the Laplace transformation to both sides of (1)

\[
\begin{align*}
\mathcal{L}\{u\} = \mathcal{L}\{g_1\} + \mathcal{L}\{R_1 (u, v, w, p) + N_1 (u, v, w, p)\}, \\
\mathcal{L}\{v\} = \mathcal{L}\{g_2\} + \mathcal{L}\{R_2 (u, v, w, p) + N_2 (u, v, w, p)\}, \\
\mathcal{L}\{w\} = \mathcal{L}\{g_3\} + \mathcal{L}\{R_3 (u, v, w, p) + N_3 (u, v, w, p)\}, \\
\mathcal{L}\{p\} = \mathcal{L}\{g_4\} + \mathcal{L}\{R_4 (u, v, w, p) + N_4 (u, v, w, p)\}.
\end{align*}
\]
(9)

Using the formulas of the Laplace transform, we get

\[
\begin{align*}
\mathcal{L}\{u\} - u(0) = \mathcal{L}\{g_1\} + \mathcal{L}\{R_1 (u, v, w, p) + N_1 (u, v, w, p)\}, \\
\mathcal{L}\{v\} - v(0) = \mathcal{L}\{g_2\} + \mathcal{L}\{R_2 (u, v, w, p) + N_2 (u, v, w, p)\}, \\
\mathcal{L}\{w\} - w(0) = \mathcal{L}\{g_3\} + \mathcal{L}\{R_3 (u, v, w, p) + N_3 (u, v, w, p)\}, \\
\mathcal{L}\{p\} - p(0) = \mathcal{L}\{g_4\} + \mathcal{L}\{R_4 (u, v, w, p) + N_4 (u, v, w, p)\}.
\end{align*}
\]
(10)

In the Laplace decomposition method, we assume the solution as an infinite series, given as follows:

\[
u = \sum_{n=0}^{\infty} u_n, \quad v = \sum_{n=0}^{\infty} v_n, \quad w = \sum_{n=0}^{\infty} w_n, \quad p = \sum_{n=0}^{\infty} p_n,
\]
(11)

where the terms \(u_n, v_n, w_n\) and \(p_n\) are to be recursively computed. In addition, the linear and nonlinear terms \(R_1, R_2, R_3, R_4\) and \(N_1, N_2, N_3, N_4\) are decompose as an infinite series of Adomian polynomials (see [12-16]):
From a numerical point of view, the approximation
\[ u(x) = \lim_{n \to \infty} [\phi_{1n}], \quad r(x) = \lim_{n \to \infty} [\phi_{2n}], \quad w(x) = \lim_{n \to \infty} [\phi_{3n}], \]
\[ p(x) = \lim_{n \to \infty} [\phi_{4n}], \]
where
\[ \phi_{1n} = \sum_{k=0}^{n-1} u_k(x), \quad \phi_{2n} = \sum_{k=0}^{n-1} v_k(x), \quad \phi_{3n} = \sum_{k=0}^{n-1} w_k(x), \quad \phi_{4n} = \sum_{k=0}^{n-1} p_k(x), \]
can be used in the Laplace decomposition scheme for computing the approximate solution. It is also clear that a better approximation can be obtained by evaluating more components of the series solution (11) of \( u(x, t), v(x, t), w(x, t) \) and \( p(x, t) \).

2.3. The Padé approximant [27]

Here we will investigate the construction of the Padé approximants [27,30–32] for the functions studied. The main advantage of Padé approximation over the Taylor series approximation is that the Taylor series expansion can exhibit oscillation, which may produce an approximation error bound. Moreover, Taylor series approximations can never blow-up in a finite region. To overcome these demerits we use the Padé approximations. The Padé approximation of a function is given by ratio of two polynomials. The coefficients of the polynomial in both the numerator and the denominator are determined using the coefficients in the Taylor series expansion of the function. The Padé approximation of a function, symbolized by \([m/n]\), is a rational function defined by
\[
[m/n] = \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m}{1 + b_1 x + b_2 x^2 + \cdots + b_n x^n},
\]
where we considered \( b_0 = 1 \), and the numerator and denominator have no common factors. In the LD–PA method, we use the method of Padé approximation as an after-treatment method to the solution obtained by the Laplace decomposition method. This after-treatment method improves the accuracy of the proposed method.

2.4. Homotopy perturbation method [33–36]

One considers the following nonlinear differential equation to represent the procedure of this method,
\[ A(U) - f(r) = 0, \quad r \in \Omega, \]
with the boundary conditions:
\[ B(U, \frac{\partial U}{\partial n}) = 0, \quad r \in \Gamma, \]
where \( A \) and \( B \) are general differential operators and boundary operator, respectively. \( \Gamma \) is the boundary of the domain \( \Omega \), and \( f(r) \) is a given analytical function. After dividing the general operator into a linear part (\( L \)) and a nonlinear part (\( N \)), one can rewrite the Eq. (15) as
\[ L(U) + N(U) - f(r) = 0. \]
By constructing the homotopy technique to Eq. (4), one can write a homotopy in the form
\[ H(V, p) = (1 - p)[L(U) - L(U_0)] + p[A(V) - f(r)] = 0, \quad p \in [0, 1], \quad r \in \Omega, \]
where \( p \in [0, 1] \) is an embedding parameter and \( U_0 \) is an initial approximation of the Eq. (15) that satisfies Eq. (16). In HPM, one can use the embedding parameter as a small parameter. Therefore, the solution of Eq. (17) can be written a power series of \( p \) in the form
\[ V = V_0 + pV_1 + p^2V_2 + \cdots. \]

Fig. 1a Exact solution of \( u(x, t) \) and numerical solution of \( u(x, t) \), \(-10 \leq x \leq 10, -1 \leq t \leq 1\).
By setting $p = 1$, one can obtain an approximate solution of Eq. (2) as,

$$U = \lim_{p \to 1} V = V_0 + V_1 + V_2 + \cdots,$$

presented in Eq. (20). The combination of a small parameter (perturbation parameter) with a homotopy is called HPM, as presented in Eq. (20).

### 3. Application

In this section, we demonstrate the analysis of all the numerical methods by applying methods to the system of partial differential Eq. (1). A comparison of all methods is also given in the form of graphs and tables, presented here.

#### 3.1. The variational iteration method [37]

To solve the system of Eq. (1), by means of variational iteration method, we construct a correctional functional which reads

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda_1 \left( u_{nt} - v_{nx} \right) dt,$$

$$v_{n+1}(x, t) = v_n(x, t) + \int_0^t \lambda_2 \left\{ \begin{array}{l} v_{nt} + 2v_{ntxx} + 6u_nv_{nx} \\ + 6v_nv_{xx} + 37w_{wxx} - 2p_{wxx} \\ - \frac{1}{2}w_{p_{xx}} + p_{p_{xx}} - 2w_{xx} - 2p_{xx} \end{array} \right\} dt,$$
where $k_1, k_2, k_3,$ and $k_4$ are general Lagrangian multipliers are to be determined [7]. With the aid of the above correction functional stationary, we obtain

$$
\lambda_1'(t) = 0, \quad \lambda_2'(t) = 0, \quad \lambda_3'(t) = 0, \quad \lambda_4'(t) = 0,
$$
$$
1 + \lambda_1(t)|_{t=0} = 0, \quad 1 + \lambda_2(t)|_{t=0} = 0, \quad 1 + \lambda_3(t)|_{t=0} = 0, \quad 1 + \lambda_4(t)|_{t=0} = 0.
$$

(21)

Eq. (21) are called Lagrange-Euler equations, one natural boundary conditions. The Lagrange multipliers, therefore, can be identified as $\lambda_1 = -1, \lambda_2 = -1, \lambda_3 = -1, \lambda_4 = -1,$ and the following variational iteration formula is given by:

$$
\begin{align*}
\nu_{n+1}(x, t) &= \nu_n(x, t) - \int_0^t \left( u_{nt} - v_{nx} \right) dt, \\
\nu_{n+1}(x, t) &= \nu_n(x, t)
\end{align*}
$$

(22)

$$
\begin{align*}
\nu_{n+1}(x, t) &= \nu_n(x, t) - \int_0^t \left\{ \nu_{nt} + 2v_{nxx} + 6u_n v_{nx} \\
&+ 6v_n u_{nx} + 37w_n w_{nx} - 2p_n w_{nx} - \frac{1}{2} w_n p_{nx} + p_n p_{nx} - 2w_{nx} - 2p_{nx} \right\} dt,
\end{align*}
$$
To solve the system of Eq. (1) by means of Laplace decomposition method, we construct a correctional functional which reads

\[
\mathcal{F}[u] = \frac{1}{s} u(0) + \frac{1}{s} \mathcal{E}[v],
\]

where \( \mathcal{E}[v] \) is the correction functional.
Applying the inverse Laplace transform, finally we get

\[
\begin{align*}
\mathcal{L}[\nu] &= \frac{1}{s}v(0) + \frac{1}{s} \mathcal{L}\left[-2v_{xxx} - 6(u)_x - 37w_x + 2pw_x\right]
+ \frac{1}{2}wp_x - pp_x + 2w_x + 2p_x, \\
\mathcal{L}[w] &= \frac{1}{s}w(0) + \frac{1}{s} \mathcal{L}[w_{xxx} + 3uw_x], \\
\mathcal{L}[p] &= \frac{1}{s}p(0) + \frac{1}{s} \mathcal{L}[p_{xxx} + 3up_x].
\end{align*}
\]

(25)

We can define the Adomian polynomial as follows:

\[
\begin{align*}
A_n &= \sum_{i=0}^{n} u_i w_{n-i}, \quad B_n &= \sum_{i=0}^{n} u_i p_{n-i}, \quad C_n &= \sum_{i=0}^{n} u_i v_{n-i}, \\
E_n &= \sum_{i=0}^{n} v_i u_{n-i}, \quad F_n &= \sum_{i=0}^{n} w_i u_{n-i}, \\
G_n &= \sum_{i=0}^{n} p_i w_{n-i}, \quad H_n &= \sum_{i=0}^{n} w_i p_{n-i}, \quad M_n &= \sum_{i=0}^{n} p_i p_{n-i},
\end{align*}
\]

we define an iterative scheme

\[
\begin{align*}
\mathcal{L}[u_{n+1}] &= \frac{1}{s} \mathcal{L}[v_n], \quad \mathcal{L}[w_{n+1}] = \frac{1}{s} \mathcal{L}[w_{xxx} + 3A_n], \\
\mathcal{L}[p_{n+1}] &= \frac{1}{s} \mathcal{L}[p_{xxx} + 3B_n], \\
\mathcal{L}[v_{n+1}] &= \frac{1}{s} \mathcal{L}[-2w_{xxx} - 6C_n - 37F_n + 2G_n] \\
&\quad + \frac{1}{2} H_n - M_n + 2w_{xx} + 2p_x, \quad n \geq 1.
\end{align*}
\]

(27)

Applying the inverse Laplace transform, finally we get

\[
\begin{align*}
u_0 &= \frac{t \sinh(x)}{\cosh(x)^2}, \quad v_1(x,t) = -\frac{1}{4} \frac{t \sinh(x)}{\cosh(x)^2}, \\
w_1(x,t) &= -\frac{1}{24} \frac{t \sinh(x)}{\cosh(x)^2}, \quad p_1(x,t) = -\frac{37}{48} \frac{t \sinh(x)}{\cosh(x)^2}, \\
w_2(x,t) &= \frac{1}{8} \frac{t^2 (2 \cosh(x)^2 - 3)}{\cosh(x)^4}, \\
v_2(x,t) &= -\frac{1}{32} \frac{t^2 (2 \cosh(x)^2 - 3)}{\cosh(x)^4}, \\
w_3(x,t) &= -\frac{1}{96} \frac{t^2 \sinh(x)}{\cosh(x)^3}, \\
p_2(x,t) &= -\frac{37}{192} \frac{t^2 \sinh(x)}{\cosh(x)^3}, \\
u_3(x,t) &= \frac{1}{24} \frac{t^3 \sinh(x) \cosh(x)^2 - 3)}{\cosh(x)^5}, \\
v_3(x,t) &= -\frac{1}{96} \frac{t^3 \sinh(x) \cosh(x)^2 - 3)}{\cosh(x)^5}, \\
w_3(x,t) &= -\frac{1}{1152} \frac{t^3 (2 \cosh(x)^2 - 3)}{\cosh(x)^5}, \\
p_3(x,t) &= -\frac{37}{2304} \frac{t^3 (2 \cosh(x)^2 - 3)}{\cosh(x)^5}, \\
u_4(x,t) &= \frac{16}{u(0)} + \frac{1}{s} \mathcal{L}\left[-2v_{xxx} - 6(u)_x - 37w_x + 2pw_x\right]
+ \frac{1}{2}wp_x - pp_x + 2w_x + 2p_x, \\
\end{align*}
\]

(28)

and so on.

Similarly, for the 16th term the solution takes the following form:

\[
\begin{align*}
u(x,t) &= \sum_{i=0}^{16} u_i(x,t), \quad p(x,t) = \sum_{i=0}^{16} p_i(x,t), \quad w(x,t) \\
&= \sum_{i=0}^{16} u_i(x,t), \quad p(x,t) = \sum_{i=0}^{16} u_i(x,t),
\end{align*}
\]

(29)

or

\[
\begin{align*}
u(x,t) &= \frac{7}{12} - 2 \tanh(x)^2 + \frac{t \sinh(x)}{\cosh(x)^2} + \frac{t^2 (2 \cosh(x)^2 - 3)}{8} \cosh(x)^4 \\
&\quad + \frac{1}{2} \frac{t^3 \sinh(x) \cosh(x)^2 - 3)}{\cosh(x)^5} \\
&\quad + \frac{1}{384} \frac{t^4 (2 \cosh(x)^2 - 15 \cosh(x)^2 + 15)}{\cosh(x)^6} \\
&\quad + \frac{1}{15360} \frac{t^5 (2 \cosh(x)^2 - 30 \cosh(x)^2 + 45)}{\cosh(x)^7} + \ldots,
\end{align*}
\]

\[
\begin{align*}
u(x,t) &= \frac{36}{37} + \frac{1}{6} \tanh(x)^2 - \frac{1}{4} \frac{t \sinh(x)}{\cosh(x)^2} - \frac{1}{96} \frac{t^2 \sinh(x)}{\cosh(x)^3}, \\
&\quad - \frac{1}{1152} \frac{t^3 (2 \cosh(x)^2 - 3)}{\cosh(x)^4} \\
&\quad - \frac{1}{4608} \frac{t^4 (2 \cosh(x)^2 - 3)}{\cosh(x)^5} \\
&\quad - \frac{1}{92160} \frac{t^5 (2 \cosh(x)^2 - 15 \cosh(x)^2 + 15)}{\cosh(x)^6} + \ldots,
\end{align*}
\]

\[
\begin{align*}
u(x,t) &= 4 + \frac{37}{12} \tanh(x) - \frac{37}{48} \frac{t \sinh(x)}{\cosh(x)^2} - \frac{37}{192} \frac{t^2 \sinh(x) \cosh(x)^2 - 3)}{\cosh(x)^3} \\
&\quad - \frac{37}{184320} \frac{t^3 (2 \cosh(x)^2 - 15 \cosh(x)^2 + 15)}{\cosh(x)^5} + \ldots.
\end{align*}
\]

(30)

And Figs. 2a and 2b shows the numerical solution of system (1) with ten terms by (LDM)

### 3.3. The Padé approximation

We use Maple to calculate the [6/4] the Padé approximant of the infinite series solution (28), which gives the following rational fraction approximation to the solution:
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\[ u(x, t) = \frac{1}{7431782400} - 7793718517201423564800000 \cosh(x)^{20} + 115359246933804711936000 \cosh(x)^{24} + \cdots, \]

\[ v(x, t) = \frac{1}{29727129600} - 7793718517201423564800000 \cosh(x)^{20} - 115359246933804711936000 \cosh(x)^{24} + \cdots, \]

\[ w(x, t) = \frac{1}{3299711385600} - 5223968936873164800 \cosh(x)^{12} \sinh(x) + 5235475476381696000 \cosh(x)^{10} \sinh(x)^{3} + 69531141601576908800 \sinh(x) \cosh(x)^{8} + \cdots, \]

\[ p(x, t) = \frac{1}{178362777600} - 5223968936873164800 \cosh(x)^{12} \sinh(x) - 5235475476381696000 \cosh(x)^{10} \sinh(x)^{3} + 69531141601576908800 \sinh(x) \cosh(x)^{8} + \cdots. \]

(31)

Fig. 3a Numerical solution of \( u(x, t) \) and \( v(x, t) \) with ten terms by (LD–PA method), \(-10 \leq x \leq 10, -1 \leq t \leq 1.\)

Fig. 3b Numerical solution of \( u(x, t) \) and \( v(x, t) \) with ten terms by (LD–PA method), \(-10 \leq x \leq 10, -1 \leq t \leq 1.\)
And the numerical solution (24) of system (1) obtained by (LD–PA) method showed in Figs. 3a and 3b, and comparison of the results obtained by the Laplace decomposition method (LDM), the variation iteration method (VIM) and the Pade approximant (LD–PA) with exact solution of system (1) presented in Figs. 4a and 4b.

3.4. Homotopy perturbation method [33]

To solve system (1) by means of HPM, we choose that:

\begin{align*}
u(x, t) &= U_0(x, t) + pU_1(x, t) + p^2 \cdot U_2(x, t) \\
&\quad + p^3 \cdot U_3(x, t) + p^4 \cdot U_4(x, t) + \cdots , \\

v(x, t) &= V_0(x, t) + p \cdot V_1(x, t) + p^2 \cdot V_2(x, t) \\
&\quad + p^3 \cdot V_3(x, t) + p^4 \cdot V_4(x, t) + \cdots , \\
w(x, t) &= W_0(x, t) + p \cdot W_1(x, t) + p^2 \cdot W_2(x, t) \\
&\quad + p^3 \cdot W_3(x, t) + p^4 \cdot W_4(x, t) + \cdots , \\
h(x, t) &= H_0(x, t) + p \cdot H_1(x, t) + p^2 \cdot H_2(x, t) \\
&\quad + p^3 \cdot H_3(x, t) + p^4 \cdot H_4(x, t) + \cdots .
\end{align*}

(32)
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We choose the initial approximations (4) and using (18) we have:

\[
p^0 : \frac{\partial}{\partial t} U_0(x, t) - \left( \frac{\partial}{\partial x} V_0(x, t) \right) = 0,
\]

\[
\frac{\partial}{\partial t} V_0(x, t) - \left( \frac{\partial}{\partial x} s_0(x, t) \right) = 0,
\]

\[
\frac{\partial}{\partial t} W_0(x, t) - \left( \frac{\partial}{\partial x} m_0(x, t) \right) = 0,
\]

\[
\frac{\partial}{\partial t} H_0(x, t) - \left( \frac{\partial}{\partial x} n_0(x, t) \right) = 0.
\]

(33)

\[
p^1 : \frac{\partial}{\partial t} U_1(x, t) + \frac{\partial}{\partial t} r_0(x, t) - \left( \frac{\partial}{\partial x} V_0(x, t) \right) = 0,
\]

\[
\frac{\partial}{\partial t} V_1(x, t) + \frac{\partial}{\partial t} s_0(x, t) + 2 \left( \frac{\partial^2}{\partial x^2} V_0(x, t) \right) - \left( \frac{\partial}{\partial x} W_0(x, t) \right)
\]

\[
- \left( \frac{\partial}{\partial x} H_0(x, t) \right) + 6 \left( \frac{\partial}{\partial x} U_0(x, t) \right) V_0(t, t)
\]

\[
+ 6 U_0(x, t) \left( \frac{\partial}{\partial x} V_0(x, t) \right) - 6 W_0(x, t) \left( \frac{\partial}{\partial x} W_0(x, t) \right)
\]

\[
- 7 H_0(x, t) \left( \frac{\partial}{\partial x} W_0(x, t) \right) - W_0(x, t) \left( \frac{\partial}{\partial x} H_0(x, t) \right)
\]

\[
- H_0(x, t) \left( \frac{\partial}{\partial x} H_0(x, t) \right) = 0,
\]

\[
\frac{\partial}{\partial t} W_1(x, t) + \frac{\partial}{\partial t} m_0(x, t) - \left( \frac{\partial^2}{\partial x^2} W_0(x, t) \right)
\]

\[
- 3 U_0(x, t) \left( \frac{\partial}{\partial x} W_0(x, t) \right)
\]

\[
- 3 U_0(x, t) \left( \frac{\partial}{\partial x} H_0(x, t) \right) = 0,
\]

\[
\frac{\partial}{\partial t} H_1(x, t) + \frac{\partial}{\partial t} n_0(x, t) - \left( \frac{\partial^3}{\partial x^3} H_0(x, t) \right)
\]

\[
- 3 U_0(x, t) \left( \frac{\partial}{\partial x} H_0(x, t) \right) = 0,
\]

(34)

Solving system (33)–(35) we have:

\[
U_1 = \frac{t \sinh(x)}{\cosh(x)^2}, \quad V_1 = -\frac{1}{4} \frac{t \sinh(x)}{\cosh(x)^2},
\]

\[
W_1 = -\frac{1}{24} \frac{t}{\cosh(x)^2}, \quad P_1 = -\frac{37}{48} \frac{t}{\cosh(x)^2},
\]

\[
U_2 = 1 \frac{t^2 (2 \cosh(x)^2 - 3)}{8 \cosh(x)^4},
\]

\[
V_2 = -\frac{1}{32} \frac{t^2 (2 \cosh(x)^2 - 3)}{\cosh(x)^4},
\]

\[
W_2 = -\frac{1}{96} \frac{t^2 \sinh(x)}{\cosh(x)^3}, \quad P_2 = -\frac{37}{192} \frac{t^2 \sinh(x)}{\cosh(x)^3},
\]

\[
U_3 = \frac{1}{24} \frac{t^3 \sinh(x) (\cosh(x)^2 - 3)}{\cosh(x)^6},
\]

\[
V_3 = -\frac{1}{96} \frac{t^3 \sinh(x) (\cosh(x)^2 - 3)}{\cosh(x)^6},
\]

\[
W_3 = -\frac{1}{1152} \frac{t^3 (2 \cosh(x)^2 - 3)}{\cosh(x)^8},
\]

\[
H_3 = -\frac{37}{2304} \frac{t^3 (2 \cosh(x)^2 - 3)}{\cosh(x)^8},
\]

\[
U_4 = \frac{1}{384} \frac{t^4 (2 \cosh(x)^4 - 15 \cosh(x)^2 + 15)}{\cosh(x)^{10}},
\]

\[
V_4 = -\frac{1}{1536} \frac{t^4 (2 \cosh(x)^4 - 15 \cosh(x)^2 + 15)}{\cosh(x)^{10}},
\]

\[
W_4 = -\frac{1}{4608} \frac{t^4 \sinh(x) (\cosh(x)^2 - 3)}{\cosh(x)^{10}},
\]

\[
H_4 = -\frac{37}{9216} \frac{t^4 \sinh(x) (\cosh(x)^2 - 3)}{\cosh(x)^{10}}.
\]

(36)

For calculating 13th term and using (20), we obtain the following solution:

\[
p^\beta : \frac{\partial}{\partial t} U_\beta(x, t) - \left( \frac{\partial}{\partial x} V_{\beta-1}(x, t) \right) = 0,
\]

\[
\frac{\partial}{\partial t} V_\beta(x, t) + 2 \left( \frac{\partial^2}{\partial x^2} V_{\beta-1}(x, t) \right) + 6 \sum_{\gamma=1}^{\beta} U_\gamma(x, t) \left( \frac{\partial}{\partial x} V_{\gamma-1}(x, t) \right)
\]

\[
+ 6 \sum_{\gamma=1}^{\beta-1} \left( \frac{\partial}{\partial x} U_\gamma(x, t) \right) V_{\gamma-1}(x, t)
\]

\[
- 6 \sum_{\gamma=1}^{\beta-1} W_\gamma(x, t) \left( \frac{\partial}{\partial x} W_{\gamma-1}(x, t) \right) - 7 \sum_{\gamma=1}^{\beta-1} H_\gamma(x, t) \left( \frac{\partial}{\partial x} H_{\gamma-1}(x, t) \right)
\]

\[
- \sum_{\gamma=1}^{\beta-1} W_\gamma(x, t) \left( \frac{\partial}{\partial x} H_{\gamma-1}(x, t) \right) - \left( \frac{\partial}{\partial x} W_{\beta-1}(x, t) \right)
\]

\[
- \left( \frac{\partial}{\partial x} H_{\beta-1}(x, t) \right) = 0,
\]

\[
\frac{\partial}{\partial t} W_\beta(x, t) - \left( \frac{\partial^3}{\partial x^3} W_{\beta-1}(x, t) \right) - 3 \sum_{\gamma=1}^{\beta-1} U_\gamma(x, t) \left( \frac{\partial}{\partial x} W_{\gamma-1}(x, t) \right)
\]

\[
- 3 \sum_{\gamma=1}^{\beta-1} U_\gamma(x, t) \left( \frac{\partial}{\partial x} H_{\gamma-1}(x, t) \right) = 0,
\]

(35)
Tables 1-4 display the absolute error obtained by (LDM), (LD-PD), (HAP) and (VIM) respectively.
Table 4 Absolute error of $p(x, t)$.

| $t$  | $| p_{ex} - p_{LDm} |$ | $| p_{ex} - u_{pade} |$ | $| p_{ex} - u_{HPM} |$ | $| p_{ex} - p_{VIM} |$ |
|------|-----------------|-----------------|-----------------|-----------------|
| 0.2  | $2.4 \times 10^{-71}$ | $6.7 \times 10^{-59}$ | $7.7 \times 10^{-64}$ | $2.3 \times 10^{-39}$ |
| 0.4  | $3.1 \times 10^{-66}$ | $1.4 \times 10^{-55}$ | $1.2 \times 10^{-59}$ | $1.9 \times 10^{-38}$ |
| 0.6  | $3.1 \times 10^{-63}$ | $1.3 \times 10^{-53}$ | $3.7 \times 10^{-57}$ | $6.5 \times 10^{-38}$ |
| 0.8  | $4.1 \times 10^{-61}$ | $3.2 \times 10^{-52}$ | $2.1 \times 10^{-55}$ | $1.5 \times 10^{-37}$ |
| 1.0  | $1.8 \times 10^{-59}$ | $3.9 \times 10^{-51}$ | $4.8 \times 10^{-54}$ | $3.1 \times 10^{-37}$ |
| 1.2  | $4.1 \times 10^{-58}$ | $3.0 \times 10^{-50}$ | $6.2 \times 10^{-53}$ | $5.4 \times 10^{-37}$ |
| 1.4  | $5.6 \times 10^{-57}$ | $1.7 \times 10^{-59}$ | $5.4 \times 10^{-52}$ | $8.7 \times 10^{-37}$ |
| 1.6  | $5.6 \times 10^{-56}$ | $7.7 \times 10^{-59}$ | $3.5 \times 10^{-51}$ | $1.3 \times 10^{-36}$ |
| 1.8  | $4.1 \times 10^{-55}$ | $2.9 \times 10^{-58}$ | $1.8 \times 10^{-50}$ | $1.8 \times 10^{-36}$ |
| 2.0  | $2.4 \times 10^{-54}$ | $9.8 \times 10^{-58}$ | $8.1 \times 10^{-50}$ | $2.6 \times 10^{-36}$ |

Fig. 5a Comparison exact and approximate solutions by LDM, (LD-PD) and HPM at $x = 1$ of $u(x, t)$ and $v(x, t)$.

Fig. 5b Comparison exact and approximate solutions by LDM, (LD-PD) and HPM at $x = 1$ of $w(x, t)$ and $p(x, t)$. 
Acknowledgments

We would like to express our gratitude to the reviewers for the careful reading of the manuscript and for their helpful advices and constructive comments.

References


