Position vectors of slant helices in Euclidean 3-space

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Abstract In this paper, position vector of a slant helix with respect to standard frame in Euclidean space $\mathbb{E}^3$ is studied in terms of Frenet equations. First, a vector differential equation of third order is constructed to determine a position vector of an arbitrary slant helix. In terms of solution, we determine the parametric representation of the slant helices from the intrinsic equations. Thereafter, we apply this method to find the parametric representation of a Salkowski curve, anti-Salkowski curve and a curve of constant precession, as examples of a slant helices, by means of intrinsic equations. © 2012 Egyptian Mathematical Society. Production and hosting by Elsevier B.V. All rights reserved.

1. Introduction

In the local differential geometry, we think of curves as a geometric set of points, or locus. Intuitively, we are thinking of a curve as the path traced out by a particle moving in $\mathbb{E}^3$. So, the investigating position vectors of the curves in a classical aim to determine behavior of the particle (curve).

Helix is one of the most fascinating curves in science and nature. Scientist have long held a fascinating, sometimes bordering on mystical obsession, for helical structures in nature. Helices arise in nano-springs, carbon nano-tubes, z-helices, DNA double and collagen triple helix, lipid bilayers, bacterial flagella in salmonella and escherichia coli, aerial hyphae in spirochetes, horns, tendrils, vines, screws, springs, helical staircases and sea shells [5,16,24]. Also we can see the helix curve or helical structures in fractal geometry, for instance hyperhelices[22]. In the field of computer aided geometric design and computer graphics, helices can be used for the tool path description, the simulation of kinematic motion or the design of highways, etc. [25]. From the view of differential geometry, a helix is a geometric curve with non-vanishing constant curvature $\kappa$ and non-vanishing constant torsion $\tau$ [3]. The helix may be called a circular helix or W-curve [10,18].

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Its known that straight line ($\kappa(s) = 0$) and circle ($\tau(s) = 0$) is degenerate-helix [12]. In fact, circular helix is the simplest three-dimensional spirals. One of the most interesting spiral example is $k$-Fibonacci spirals. These curves appear naturally from studying the $k$-Fibonacci numbers $\{F_k(s)\}_{n=0}^\infty$ and the related hyperbolic $k$-Fibonacci function [7]. Three-dimensional $k$-Fibonacci spirals was studied from a geometric point of view in [8].

Indeed a helix is a special case of the general helix. A curve of constant slope or general helix in Euclidean 3-space $\mathbb{E}^3$ is...
defined by the property that the tangent makes a constant angle with a fixed straight line called the axis of the general helix. A classical result stated by Lancret in 1802 and first proved by de Saint Venant in 1845 (see [21] for details) says that: A necessary and sufficient condition that a curve be a general helix is that the ratio \( \frac{\kappa}{\tau} \) is constant along the curve, where \( \kappa \) and \( \tau \) denote the curvature and the torsion, respectively. A general helices or inclined curves are well known curves in classical differential geometry of space curves [1,3,17,20,23].

Izumiya and Takeuchi [11] have introduced the concept of slant helix from intrinsic equations in Euclidean space [6,15]. This problem is not easy to solve in general case. However, this problem is solved in three special cases only. Firstly, in the case of a plane curve (\( \tau = 0 \)). Secondly, in the case of a helix (\( \kappa \) and \( \tau \) are both non-vanishing constant). Recently, Ali [2] adapted fundamental existence and uniqueness theorem for space curves in Euclidean space \( \mathbb{E}^3 \) and constructed a vector differential equation to solve this problem in the case of a general helix (\( \frac{\kappa}{\tau} \) is constant). However, this problem is not solved in other cases of the space curve.

In the light of our main problem, first we give:

**Theorem 3.1.** Let \( \psi = \psi(s) \) be a unit speed curve. Suppose \( \psi = \psi(\theta) \) is another parametric representation of this curve by the parameter \( \theta = \int \kappa(s)ds \). Then, the principal normal vector \( \mathbf{N} \) satisfies a vector differential equation of third order as follows:

\[
\frac{1}{f(\theta)} \left[ \frac{1}{f(\theta)} \left( \mathbf{N}''(\theta) + (1 + f^2(\theta))\mathbf{N}(\theta) \right) \right]' + \mathbf{N}(\theta) = 0, \tag{2}
\]

where \( f(\theta) = \frac{\kappa(\theta)}{\kappa(\theta)} \).

**Proof.** Let \( \psi = \psi(s) \) be an unit speed curve. If we write this curve in the another parametric representation \( \psi = \psi(\theta) \), where \( \theta = \int \kappa(s)ds \), we have new Frenet equations as follows:

\[
\begin{bmatrix}
\mathbf{T}'(\theta) \\
\mathbf{N}'(\theta) \\
\mathbf{B}'(\theta)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 \\
-1 & 0 & f(\theta) \\
0 & -f(\theta) & 0
\end{bmatrix}
\begin{bmatrix}
\mathbf{T}(\theta) \\
\mathbf{N}(\theta) \\
\mathbf{B}(\theta)
\end{bmatrix}, \tag{3}
\]

where \( f(\theta) = \frac{\kappa(\theta)}{\kappa(\theta)} \). If we differentiate the second equation of the new Frenet Eq. (3) and using the first and the third equations, we have

\[
\mathbf{B}(\theta) = \frac{1}{f(\theta)} [\mathbf{N}'(\theta) + (1 + f^2(\theta))\mathbf{N}(\theta)]. \tag{4}
\]

Differentiating the above equation and using the last equation from (3), we obtain a vector differential equation of third order (2) as desired. \( \square \)

The Eq. (2) is not easy to solve in general case. If one solves this equation, the natural representation of the position vector of an arbitrary space curve can be determined as follows:

\[
\psi(s) = \int \left( \int \kappa(s)\mathbf{N}(s)ds \right) ds + C, \tag{5}
\]

or in parametric representation

\[
\psi(\theta) = \int \frac{1}{\kappa(\theta)} \left( \int \mathbf{N}(\theta)d\theta \right) d\theta + C, \tag{6}
\]
where \( \theta = \int \kappa(s) ds \).

We can solve the Eq. (2) in the case of a slant helix. The following proposition are new characterizations for a slant helices in \( \mathbb{E}^3 \):

**Lemma 3.2.** Let \( \psi: I \rightarrow \mathbb{E}^3 \) be a curve that is parameterized by arclength with intrinsic equations \( \kappa = \kappa(s) \) and \( \tau = \tau(s) \). The curve is a slant helix (its normal vectors make a constant angle, \( \phi = \pm \arccos[n] \), with a fixed straight line in the space) if and only if

\[
\tau(s) = \pm \frac{m \kappa(s)}{\sqrt{1 - m^2 (\int \kappa(s) ds)^2}},
\]

where \( m = \frac{\phi}{\sqrt{1 - \phi^2}} \).

**Proof.** (\( \Rightarrow \)) Let \( \mathbf{d} \) be a unit fixed vector makes a constant angle, \( \phi = \pm \arccos[n] \), with the normal vector \( \mathbf{N} \). Therefore

\[
(N, d) = n.
\]

Differentiating the Eq. (8) with respect to the variable \( \theta = \int \kappa(s) ds \) and using the new Frenet Eq. (3), we get

\[
-\mathbf{T}(\theta) + f(\theta) \mathbf{B}(\theta), \mathbf{d} = 0.
\]

Therefore,

\[
(\mathbf{T}, \mathbf{d}) = f(B, \mathbf{d}).
\]

If we put \((B, \mathbf{d}) = b\), we can write

\[
\mathbf{d} = f b \mathbf{T} + n \mathbf{N} + b \mathbf{B}.
\]

From the unitary of the vector \( \mathbf{d} \) we get \( b = \pm \sqrt{\frac{n^2}{1 + f^2}} \). Therefore, the vector \( \mathbf{d} \) can be written as

\[
\mathbf{d} = \pm f \left( \sqrt{\frac{1 - n^2}{1 + f^2}} \mathbf{T} + n \mathbf{N} \right) \pm \left( \sqrt{\frac{1 - n^2}{1 + f^2}} \mathbf{B} \right).
\]

If we differentiate Eq. (9) again, we obtain

\[
\left(f' B - (1 + f^2)N, d\right) = 0.
\]

Eqs. (10) and (11) lead to the following differential equation

\[
f'(1 + f^2)^{3/2} = \pm m,
\]

where \( m = \frac{\phi}{\sqrt{1 - \phi^2}} \). Integration the above equation, we get

\[
f(1 + f^2) = \pm m(\theta + c_1).
\]

where \( c_1 \) is an integration constant. The integration constant can disappear with a parameter change \( \theta \rightarrow \theta - c_1 \). Solving the Eq. (12) with \( f \) as unknown we have

\[
f(\theta) = \pm \frac{m \theta}{\sqrt{1 - m^2 \theta^2}}.
\]

Finally, \( \tau(s) = \kappa(s) f(s) \), we express the desired result.

(\( \Leftarrow \)) Suppose that \( \tau(s) = \pm \frac{m \kappa(s)}{\sqrt{1 - m^2 (\int \kappa(s) ds)^2}} \). The function \( f \) can be written as \( f(\theta) = \pm \frac{m \theta}{\sqrt{1 - m^2 \theta^2}} \) and let us consider the vector

\[
\mathbf{d} = n \left( \mathbf{T} + \frac{1}{m} \mathbf{N} \right) \pm \frac{1}{m} \mathbf{B}.
\]

We will prove that the vector \( \mathbf{d} \) is a constant vector. Indeed, applying Frenet formula (3)

\[
\mathbf{d'} = n \left( \mathbf{T} + \theta \mathbf{N} - \mathbf{T} + \frac{m \theta}{\sqrt{1 - m^2 \theta^2}} \mathbf{B} \right) + \frac{L}{m} \mathbf{N} = 0.
\]

Therefore, the vector \( \mathbf{d} \) is constant and \((N, d) = n\). This concludes the proof of Lemma 3.2. \( \square \)

**Theorem 3.3.** The position vector \( \psi = (\psi_1, \psi_2, \psi_3) \) of a slant helix is computed in the natural representation form:

\[
\begin{align*}
\psi_1(s) &= \frac{m}{m} \int \frac{\kappa(s)}{\sqrt{1 - m^2 (\int \kappa(s) ds)^2}} \cos \left( \frac{1}{m} \arcsin \left( m f(\kappa(s) ds) \right) \right) ds, \\
\psi_2(s) &= \frac{n}{m} \int \frac{\kappa(s)}{\sqrt{1 - m^2 (\int \kappa(s) ds)^2}} \sin \left( \frac{1}{m} \arcsin \left( m f(\kappa(s) ds) \right) \right) ds, \\
\psi_3(s) &= n \int \frac{f(\kappa(s) ds)}{m} ds,
\end{align*}
\]

or in the parametric form:

\[
\begin{align*}
\psi_1(\theta) &= \frac{m}{m} \int \frac{\cos(\theta)}{\sqrt{1 - m^2 (\int \kappa(s) ds)^2}} \left( \cos \left( \frac{1}{m} \arcsin \left( m f(\kappa(s) ds) \right) \right) \right) d\theta, \\
\psi_2(\theta) &= \frac{n}{m} \int \frac{\sin(\theta)}{\sqrt{1 - m^2 (\int \kappa(s) ds)^2}} \left( \sin \left( \frac{1}{m} \arcsin \left( m f(\kappa(s) ds) \right) \right) \right) d\theta, \\
\psi_3(\theta) &= n \int \frac{f(\kappa(s) ds)}{m} d\theta,
\end{align*}
\]

or in the useful parametric form:

\[
\begin{align*}
\psi_1(t) &= \frac{m}{m} \int \frac{\cos(\theta)}{\sqrt{1 - m^2 (\int \kappa(s) ds)^2}} \left( \cos \left( \frac{1}{m} \arcsin \left( m f(\kappa(s) ds) \right) \right) \right) dt, \\
\psi_2(t) &= \frac{n}{m} \int \frac{\sin(\theta)}{\sqrt{1 - m^2 (\int \kappa(s) ds)^2}} \left( \sin \left( \frac{1}{m} \arcsin \left( m f(\kappa(s) ds) \right) \right) \right) dt, \\
\psi_3(t) &= \frac{\arctan(\theta)}{\sqrt{1 - m^2 (\int \kappa(s) ds)^2}} dt,
\end{align*}
\]

where \( \theta = \int \kappa(s) ds \), \( t = \frac{1}{m} \arcsin(m \theta) \), \( m = \frac{\phi}{\sqrt{1 - \phi^2}} \), \( \phi \) is the angle between the fixed straight line (axis of a slant helix) and the principal normal vector of the curve.

**Proof.** If \( \psi \) is a slant helix whose principal normal vector \( \mathbf{N} \) makes an angle \( \phi = \pm \arccos[n] \) with a straight line \( U \), then we can write \( f(\theta) = \pm \frac{m \theta}{\sqrt{1 - m^2 \theta^2}} \), where \( f = \frac{\phi}{\sqrt{1 - \phi^2}} \). Therefore, the Eq. (2) becomes

\[
(1 - m^2 \theta^2) \mathbf{N}'(\theta) - 3m^2 \theta^2 \mathbf{N}'(\theta) + \mathbf{N}'(\theta) = 0.
\]

If we write the principal normal vector as the following:

\[
\mathbf{N} = N_1(\theta) \mathbf{e}_1 + N_2(\theta) \mathbf{e}_2 + N_3(\theta) \mathbf{e}_3.
\]

Now, the curve \( \psi \) is a slant helix, i.e. the principal normal vector \( \mathbf{N} \) makes a constant angle, \( \phi \), with the constant vector called the axis of the slant helix. So, without loss of generality, we take the axis of a slant helix parallel to \( \mathbf{e}_3 \). Then

\[
N_3(\theta) = (N, e_3) = n.
\]

On the other hand the principal normal vector \( \mathbf{N} \) is a unit vector, so the following condition is satisfied

\[
N_1(\theta) + N_2(\theta) = 1 - n^2 = \frac{n^2}{m^2}.
\]

The general solution of Eq. (20) can be written in the following form:

\[
N_1(\theta) + N_2(\theta) = 1 - n^2 = \frac{n^2}{m^2},
\]

where \( t \) is an arbitrary function of \( \theta \). Every component of the vector \( \mathbf{N} \) is satisfied the Eq. (17). So, substituting the compo-
nents $N_1(\theta)$ and $N_2(\theta)$ in the Eq. (17), we have the following differential equations of the function $t(\theta)$
\[3'(m^2\theta ' - (1 - m^2\theta^2))\cos[t] - \left(t' - 3m^2\theta t' - (1 - m^2\theta^2)(t^2 - t''')\right)\sin[t] = 0, \tag{22}\]
\[3'(m^2\theta ' - (1 - m^2\theta^2))\sin[t] + \left(t' - 3m^2\theta t' - (1 - m^2\theta^2)(t^2 - t''')\right)\cos[t] = 0. \tag{23}\]

It is easy to prove that the above two equations lead to the following two equations:
\[m^2\theta ' - (1 - m^2\theta^2)\theta' = 0, \tag{24}\]
\[t' - 3m^2\theta t' - (1 - m^2\theta^2)(t^2 - t''') = 0. \tag{25}\]

The general solution of Eq. (24) is
\[t(\theta) = c_2 + c_1 \arcsin(m\theta), \tag{26}\]
or
\[t(\theta) = c_2 + c_1 \arccos(m\theta), \tag{27}\]
where $c_1$ and $c_2$ are constants of integration. The constant $c_2$ can be disappear if we change the parameter $t \rightarrow t + c_2$.

Substituting the solution (26) or (27) in the Eq. (25), we obtain the following condition:
\[m c_1 (1 + m^2(1 - c_1)) = 0\]
which leads to $c_1 = -\sqrt{1 + m^2} = \frac{1}{2}$, where $m \neq 0$ and $c_1 \neq 0$.

Now, the principal normal vector take the following form:
\[N(\theta) = \left( -\frac{n}{m}\frac{n}{m} \cos \left[ \frac{1}{n} \arcsin(m\theta) \right], \frac{n}{m} \frac{n}{m} \sin \left[ \frac{1}{n} \arcsin(m\theta) \right], n \right). \tag{28}\]
or
\[N(\theta) = \left( -\frac{n}{m}\frac{n}{m} \cos \left[ \frac{1}{n} \arccos(m\theta) \right], \frac{n}{m} \frac{n}{m} \sin \left[ \frac{1}{n} \arccos(m\theta) \right], n \right). \tag{29}\]

If we substitute the Eq. (28) in the two Eqs. (5) and (6), we have the two Eqs. (14) and (15). It is easy to arrive the Eq. (16), if we take the new parameter $t = \frac{1}{n} \arcsin(m\theta)$, which completes the proof.

On the other hand if we used Eq. (29), we have the following theorem:

**Theorem 3.4.** The position vector $\psi = (\psi_1, \psi_2, \psi_3)$ of a slant helix is given in the natural representation form:
\[\begin{align*}
\psi_1(s) &= \frac{\sqrt{m}}{m} \int \left[ \frac{1}{n} \arcsin(m\theta) \right] d\theta, \\
\psi_2(s) &= \frac{\sqrt{m}}{m} \int \left[ \frac{1}{n} \arcsin(m\theta) \right] ds, \\
\psi_3(s) &= n \int \left[ \frac{1}{n} \arcsin(m\theta) \right] ds, \tag{30}\end{align*}\]
or in the parametric form:
\[\begin{align*}
\psi_1(\theta) &= \frac{\sqrt{m}}{m} \int \left[ \frac{1}{n} \arcsin(m\theta) \right] d\theta, \\
\psi_2(\theta) &= \frac{\sqrt{m}}{m} \int \left[ \frac{1}{n} \arcsin(m\theta) \right] d\theta, \\
\psi_3(\theta) &= n \int \left[ \frac{1}{n} \arcsin(m\theta) \right] d\theta. \tag{31}\end{align*}\]

or in the useful parametric form:
\[\begin{align*}
\psi_1(t) &= \frac{\sqrt{m}}{m} \int \left[ \frac{1}{n} \arcsin(m\theta) \right] \sin[m\theta] dt, \\
\psi_2(t) &= \frac{\sqrt{m}}{m} \int \left[ \frac{1}{n} \arcsin(m\theta) \right] \cos[m\theta] dt, \\
\psi_3(t) &= -\frac{\sqrt{m}}{m} \int \left[ \frac{1}{n} \arcsin(m\theta) \right] \cos[m\theta] dt, \tag{32}\end{align*}\]
where $\theta = \int \kappa(s) ds$, $t = \frac{1}{m} \arcsin(m\theta)$, $m = \frac{1}{2} n \arcsin[\phi]$ and $\phi$ is the angle between the fixed straight line (axis of a slant helix) and the principal normal vector of the curve.

**4. Examples**

In this section, we take several choices for the curvature $\kappa$ and torsion $\tau$, and next, we apply Theorem 3.3.

**Example 4.1.** The case of a slant helix with $\kappa = 1$, $\tau = \frac{m s}{\sqrt{1 - m^2 s^2}}$. \tag{33}

which are the intrinsic equations of a Salkowski curve \[18\]. Substituting $\kappa(t) = 1$ in the Eq. (16) we have the explicit parametric representation of such curve as follows:
\[\begin{align*}
\psi_1(t) &= \frac{\sqrt{m}}{m} \int \left[ \frac{1}{n} \arcsin(m\theta) \right] \sin[(2n + 1)t] + \frac{\sqrt{m}}{2m} \cos[(2n - 1)t] - 2 \cos[t], \\
\psi_2(t) &= \frac{\sqrt{m}}{m} \int \left[ \frac{1}{n} \arcsin(m\theta) \right] \cos[(2n + 1)t] + \frac{\sqrt{m}}{2m} \sin[(2n - 1)t] - 2 \sin[t], \\
\psi_3(t) &= -\frac{\sqrt{m}}{m} \int \left[ \frac{1}{n} \arcsin(m\theta) \right] \cos[2nt]. \tag{34}\end{align*}\]

where $t = \frac{1}{m} \arcsin(m\theta)$. One can see a special examples of such curves in the Fig. 1.

**Example 4.2.** The case of a slant helix with $\kappa = \frac{m s}{\sqrt{1 - m^2 s^2}}$, $\tau = 1$. \tag{35}

which are the intrinsic equations of an anti-Salkowski curve \[18\]. Substituting
\[\kappa = \frac{m s}{\sqrt{1 - m^2 s^2}} = \frac{1}{m} \cot[nt], \]
in the Eq. (16) we have the explicit parametric representation of such curve as follows:
\[\begin{align*}
\psi_1(t) &= \frac{\sqrt{m}}{m} \int \left[ \frac{1}{n} \arcsin(m\theta) \right] \sin[(1 + 2nt)t] + \frac{\sqrt{m}}{2m} \cos[(1 - 2nt)t] + 2n \cos[t], \\
\psi_2(t) &= \frac{\sqrt{m}}{m} \int \left[ \frac{1}{n} \arcsin(m\theta) \right] \cos[(1 + 2nt)t] + \frac{\sqrt{m}}{2m} \sin[(1 - 2nt)t] - 2n \sin[t], \\
\psi_3(t) &= \frac{\sqrt{m}}{m} \int \left[ \frac{1}{n} \arcsin(m\theta) \right] (2nt - \sin[2nt]). \tag{36}\end{align*}\]

where $\theta = \frac{\sqrt{m^2 s^2}}{m}$ and $t = \frac{1}{m} \arcsin(m\theta)$. One can see a special examples of such curves in the Fig. 2.

**Remark 4.3.** A family of curves with constant curvature but non-constant torsion is called Salkowski curves and a family of curves with constant torsion but non-constant curvature is called anti-Salkowski curves \[19\]. Monterde [18] studied some of characterizations of these curves and he proved that the principal normal vector makes a constant angle with fixed straight line. So that: Salkowski and anti-Salkowski curves are important examples of slant helices.
Example 4.4. The case of a slant helix with
\[ \kappa = \frac{\mu}{m} \cos[\mu \, s], \quad \tau = \frac{\mu}{m} \sin[\mu \, s]. \] (37)

Substituting \( \kappa = \frac{\mu}{m} \cos[\mu \, s] \) in the Eq. (14) we have the natural
representation of such curve as follows:
\[
\begin{align*}
\psi_1(s) &= -\frac{\mu}{\pi^2} [ (1 + n^2) \cos[\mu \, s] \cos[\frac{\pi n}{\mu}] + 2 n \sin[\mu \, s] \sin[\frac{\pi n}{\mu}] ], \\
\psi_2(s) &= -\frac{\mu}{\pi^2} [ (1 + n^2) \cos[\mu \, s] \sin[\frac{\pi n}{\mu}] - 2 n \sin[\mu \, s] \cos[\frac{\pi n}{\mu}] ], \\
\psi_3(s) &= -\frac{\mu}{\pi^2} \cos[\mu \, s].
\end{align*}
\] (38)

The above curve is a geodesic of the tangent developable of a general helix [11]. One can see a special examples of such curves in the Fig. 3.

Remark 4.5. A unit speed curve of constant precession is defined by the property that its (Frenet) Darboux vector
\[ W = \tau \, T + \kappa \, B \]
revolves about a fixed line in space with angle and constant speed. A curve of constant precession is characterized by having

\[
\begin{align*}
\psi_1(s) &= -\frac{\mu}{\pi^2} [ (1 + n^2) \cos[\mu \, s] \cos[\frac{\pi n}{\mu}] + 2 n \sin[\mu \, s] \sin[\frac{\pi n}{\mu}] ], \\
\psi_2(s) &= -\frac{\mu}{\pi^2} [ (1 + n^2) \cos[\mu \, s] \sin[\frac{\pi n}{\mu}] - 2 n \sin[\mu \, s] \cos[\frac{\pi n}{\mu}] ], \\
\psi_3(s) &= -\frac{\mu}{\pi^2} \cos[\mu \, s].
\end{align*}
\] (38)
\[ \kappa = \frac{\mu}{m} \sin[\mu s], \quad \tau = \frac{\mu}{m} \cos[\mu s] \]

or

\[ \kappa = \frac{\mu}{m} \cos[\mu s], \quad \tau = \frac{\mu}{m} \sin[\mu s] \]

where \( \mu \) and \( m \) are constants. This curve lie on a circular one-sheeted hyperboloid

\[ x^2 + y^2 - m^2 z^2 = 4m^2. \]

The curve closed if and only if \( n = \frac{m}{\sqrt{1+m^2}} \) is rational [20]. Kula and Yayli [13] have proved that the geodesic curvature of the spherical image of the principal normal indicatrix of a curve of constant precession is a constant function equal \(-m\). So that: the curves of constant precessions are important examples of slant helices.

The curves which considered in examples (4.1), (4.2) and (4.4) are platted in Figs. 1–3, respectively.

References